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The method of Poisson pairs in the theory of nonlinear PDEs[1]

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Abstract

The aim of these lectures is to show that the methods of classical Hamiltonian mechanics can be profitably used to solve certain classes of nonlinear partial differential equations. The prototype of these equations is the well-known Korteweg–de Vries (KdV) equation.
In these lectures we touch the following subjects:

i) the birth and the role of the method of Poisson pairs inside the theory of the KdV equation;

ii) the theoretical basis of the method of Poisson pairs;

iii) the Gel'fand–Zakharevich theory of integrable systems on bi-Hamiltonian manifolds;

iv) the Hamiltonian interpretation of the Sato picture of the KdV flows and of its linearization on an infinite–dimensional Grassmannian manifold.

v) the reduction technique(s) and its use to construct classes of solutions;

vi) the role of the technique of separation of variables in the study of the reduced systems;

vii) some relations intertwining the method of Poisson pairs with the method of Lax pairs.

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Contents

1 Introduction: The tensorial approach and the birth of the method of Poisson pairs

This lecture is an introduction to the Hamiltonian analysis of PDEs form an "experimental" point of view. This means that we are more concerned in unveiling the spirit of the method than in working out the theoretical details. Therefore the style of the exposition will be informal, and proofs will be mainly omitted. We shall follow, step by step, the birth and the evolution of the Hamiltonian analysis of the KdV equation

$$u_t = \frac{1}{4}(u_{xxx} - 6uu_x) \ , \tag{1.1}$$

from its "infancy" to the final representation of the KdV flow as a linear flow on an infinite-dimensional Grassmannian due to Sato [27]. The route is long and demanding. Therefore the exposition is divided in two parts, to be carried out in this and in the fourth lecture (see Section 4). Here our primary aim is to show the birth of the method of Poisson pairs. It is reached by means of a suitable use of the well-known methods of tensor analysis. We proceed in three steps. First, by using the transformation laws of vector fields, we construct the Miura map and the so called *modified* KdV equation (mKdV). This result leads quite simply to the theory of (elementary) Darboux transformations and to the concept of Poisson pair. Indeed, a peculiarity of mKdV is to possess an elementary Hamiltonian structure. By means of the transformation law of Poisson bivectors, we are then able to transplant this structure to the KdV equation, unraveling its "bi-Hamiltonian structure". This structure can be used in turn to define the concept of *Lenard chain* and to plunge the KdV equation into the "KdV hierarchy". This step is rather important from the point of view of finding classes of solutions to the KdV equation. Indeed the hierarchy is a powerful instrument to construct finite-dimensional invariant submanifolds of the equation and, therefore, finite-dimensional reductions of the KdV equation. The study of this process of restriction and of its use to construct solutions will be one of the two leading themes of these lectures. It is intimately related to the theory of separation of variables dealt with in the last two lectures. The second theme is that of the linearization of the full KdV flow on the infinite-dimensional Sato Grassmannian. The starting point of this process is surprisingly simple, and once again based on a simple procedure of tensor calculus. By means of the transformation laws of one–forms, we pull back the KdV hierarchy from its phase space onto the phase space of the mKdV equation. In this way we obtain the "mKdV hierarchy". In the fourth lecture we shall show that this hierarchy can be written as a flow on an infinite–dimensional Grassmann manifold, and that this flow can be linearized by means of a (generalized) Darboux transformation.

1.1 The Miura map and the KdV equation

As an effective way of probing the properties of equation (1.1) we follow the tensorial approach. Accordingly, we regard equation (1.1) as the *definition* of a

vector field

$$u_t = X(u, u_x, u_{xx}, u_{xxx}) \tag{1.2}$$

on a suitable function space, and we investigate how it transforms under a point transformation in this space. Since our "coordinate" u is a function and not simply a number, we are allowed to consider transformations of coordinates depending also on the derivatives of the new coordinate function of the type[2]

$$u = \Phi(h, h_x) \ . \tag{1.3}$$

We ask whether there exists a transformed vector field

$$h_t = Y(h, h_x, h_{xx}, h_{xxx}) \tag{1.4}$$

related to the KdV equation according to the transformation law for vector fields,

$$X(\Phi(h)) = \Phi'_h \cdot Y(h), \tag{1.5}$$

where Φ'_h is the (Fréchet) derivative of the operator Φ defining the transformation. This condition gives rise to a (generally speaking) over-determined system of partial differential equations on the unknown functions $\Phi(h, h_x)$ and $Y(h, h_x, h_{xx}, h_{xxx})$. In the specific example the over-determined system can be solved. Apart form the trivial solution $u = h_x$, we find the *Miura transformation* [23, 16],

$$u = h_x + h^2 - \lambda \ , \tag{1.6}$$

depending on an arbitrary parameter λ. The transformed equation is the *modified* KdV equation:

$$h_t = \frac{1}{4}(h_{xxx} - 6h^2 h_x + 6\lambda h_x) \ . \tag{1.7}$$

Exercise 1.1 Work out in detail the transformation law (1.5), checking that X, Φ, and Y, defined respectively by equations (1.1), (1.6) and (1.7) do satisfy equation (1.5). □

The above result is plenty of consequences. The first one is a simple method for constructing solutions of the KdV equation. It is called the method of (elementary) Darboux transformations [22]. It rests on the remark that the mKdV equation (1.7) admits the discrete symmetry

$$h \mapsto h' = -h \ . \tag{1.8}$$

Let us exploit this property to construct the well-known one–soliton solution of the KdV equation. We notice that the point $u = 0$ is a very simple (singular)

[2] For further details on these kind of transformations, see [8].

invariant submanifold of the KdV equation. Its inverse image under the Miura transformation is the 1–dimensional submanifold S_1 formed by the solutions of the special Riccati equation

$$h_x + h^2 = \lambda \ . \tag{1.9}$$

This submanifold, in its turn, is invariant with respect to equation (1.7). A straightforward computation shows that, on this submanifold,

$$\frac{1}{4}(h_{xxx} - 6h^2 h_x + 6\lambda h_x) = \lambda h_x \ . \tag{1.10}$$

Therefore, on S_1 the mKdV equation takes the simple form $h_t = \lambda h_x$. Solving the first order system formed by this equation and the the Riccati equation $h_x + h^2 = \lambda$, and setting $\lambda = z^2$, we find the general solution

$$h(x,t) = z\ \tanh(zx + z^3 t + c) \tag{1.11}$$

of the mKdV equation on the invariant submanifold S_1. At this point we use the symmetry property and the Miura map. By the symmetry property (1.8) the function $-h(x,t)$ is a new solution of the modified equation, and by the Miura map the function

$$u'(x,t) = -h_x + h^2 - z^2 = 2z^2 \mathrm{sech}^2(zx + z^3 t + c) \tag{1.12}$$

is a new solution of the KdV equation. It is called the *one soliton* solution[3]. It can also be interpreted in terms of invariant submanifolds. To this end, we have to notice that the Miura map (1.6) transforms the invariant submanifold S_1 of the modified equation into the submanifold formed by the solutions of the first order differential equation

$$\frac{1}{2}\left(-\frac{1}{2}u_x^2 + u^3\right) + \lambda u^2 = 0 \ . \tag{1.13}$$

As one can easily check, this set is preserved by the KdV equation, and therefore it is an invariant one–dimensional submanifold of the KdV equation, built up from the singular manifold $u = 0$. On this submanifold, the KdV equation takes the simple form $u_t = \lambda u_x$, and the flow can be integrated to recover the solution (1.12).

This example clearly shows that the Darboux transformations are a mechanism to build invariant submanifolds of the KdV equation. Some of these submanifolds will be examined in great detail in the present lectures. The purpose is to show that the reduced equations on these submanifolds are classical Hamiltonian vector fields whose associated Hamilton–Jacobi equations can be solved by separation of variables. In this way, we hope, the interest of the Hamiltonian analysis of the KdV equations can better emerge.

[3]For a very nice account of the origin and of the properties of the KdV equation and of other soliton equations and their solutions, see, e.g., [24].

1.2 Poisson pairs and the KdV hierarchy

We shall now examine a more deep and far reaching property of the Miura map. It is connected with the concept of Hamiltonian vector field. From Analytical Mechanics, we know that the Hamiltonian vector fields are the images of exact one–forms through a suitable linear map, associated with a so–called Poisson bivector. We shall formally define these notions in the next lecture. These definition can be easily extended to vector fields on infinite–dimensional manifolds. Let us give an example, by showing that the mKdV equation is a Hamiltonian vector field. This requires a series of three consecutive remarks. First we notice that equation (1.7) can be factorized as

$$h_t = \left[\frac{1}{2}\partial_x\right] \cdot \left[\frac{1}{2}h_x x - \frac{3}{2}h^3 + 3\lambda h\right] . \tag{1.14}$$

Then, we notice that the linear operator in the first bracket, $\frac{1}{2}\partial_x$, is a constant skewsymmetric operator which we can recognize as a Poisson bivector. Finally, we notice that in the differential polynomial appearing in the second bracket in the right hand side of equation (1.14), we can easily recognize an exact one–form. Indeed,

$$\int \left(\frac{1}{2}h_{xx} - \frac{3}{2}h^3 + 3\lambda h\right) \dot{h}\, dx = \frac{d}{dt}\int \left(-\frac{1}{2}h_x^2 - \frac{3}{8}h^4 + \frac{3}{2}\lambda h^2\right) dx \tag{1.15}$$

for any tangent vector $\dot{h}$. These statements are true under suitable boundary conditions, as explained in, e.g., [8]. Here and in the rest of these lectures we will tacitly use periodic boundary conditions.

The Hamiltonian character of the mKdV equation is obviously independent of the existence of the Miura map. However, this map finely combines this property from the point of view of tensor analysis. Let us recall that a Poisson bivector is a skewsymmetric linear map from the cotangent to the tangent spaces satisfying a suitable differential condition (see Lecture 2). It obeys the transformation law

$$Q_{\Phi(h)} = \Phi'_h P_h \Phi'^*_h \tag{1.16}$$

under a change of coordinates (or a map between two different manifolds). In this formula the point transformation is denoted (in operator form) by $u = \Phi(h)$. The operators Φ'_h and Φ'^*_h are the Fréchet derivative of Φ and its adjoint operator. The symbols P_h and Q_u denote the Poisson bivectors in the space of the functions h and u, respectively. Since the Miura map $u = h_x + h^2 - \lambda$ is not invertible, it is rather nontrivial that there exists a Poisson bivector Q_u, on the phase space of the KdV equation, which is Φ–related (in the sense of equation (1.16)) to the Poisson bivector $P_h = \frac{1}{2}\partial_x$ associated with the modified equation. Surprisingly, this is the case. One can check that the operator Q_u is defined by

$$Q_u = -\frac{1}{2}\partial_{xxx} + 2(u+\lambda)\partial_x + u_x . \tag{1.17}$$

Exercise 1.2 Verify the above claim by computing the product (in the appropriate order) of the operators $\Phi'_h = \partial_x + h$, $P_h = \frac{1}{2}\partial_x$, and $\Phi'^*_h = -\partial_x + h$, and by expressing the results in term of $u = h_x + h^2 - \lambda$. □

This exercise shows that the Miura map is a peculiar Poisson map. Since it depends on the parameter λ, the final result is that the phase space of the KdV equation is endowed with a one–parameter family of Poisson bivectors,

$$Q_\lambda = Q_1 - \lambda Q_0 \ , \tag{1.18}$$

which we call a Poisson pencil. The operators (Q_1, Q_0) defining the pencil are said to form a *Poisson pair*, a concept to be systematically investigated in the next lecture.

These operators enjoy a number of interesting properties, and define new geometrical structures associated with the equation. One of the simplest but far–reaching is the concept of *Lenard chain*. The idea is to use the pair of bivectors to define a recursion relation on one–forms:

$$Q_0 \alpha_{j+1} = Q_1 \alpha_j \ . \tag{1.19}$$

In the applications a certain care must be taken in dealing with this recursion relation, since it does not define uniquely the forms α_j (the operator Q_0 is seldom invertible). Furthermore, it is still less apparent that it can be solved in the class of *exact* one–forms. However, in the KdV case we *bonafide* proceed and we find

$$\begin{aligned}
\alpha_0 &= 1 \\
\alpha_1 &= -\frac{1}{2}u \\
\alpha_2 &= \frac{1}{8}(-u_{xx} + 3u^2) \\
\alpha_3 &= \frac{1}{32}(-10u^3 + 10\,uu_{xx} - u_{xxxx} + 5u_x^2)
\end{aligned} \tag{1.20}$$

as first terms of the recurrence. The next step is to consider the associated vector fields (the meaning of numbering them with odd integers will be explained in Lecture 4):

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= Q_1\alpha_0 = Q_0\alpha_1 = u_x \\
\frac{\partial u}{\partial t_3} &= Q_1\alpha_1 = Q_0\alpha_2 = \frac{1}{4}(u_{xxx} - 6uu_x) \\
\frac{\partial u}{\partial t_5} &= Q_1\alpha_2 = Q_0\alpha_3 = \frac{1}{16}(u_{xxxxx} - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x) \ .
\end{aligned} \tag{1.21}$$

They are the first members of the KdV hierarchy. In the fourth lecture, we shall show that it is a special instance of a general concept, the Gel'fand–Zakharevich hierarchy associated with any Poisson pencil of a suitable class.

1.3 Invariant submanifolds and reduced equations

The introduction of the KdV hierarchy has important consequences on the problem of constructing solutions of the KdV equation. The hierarchy is indeed a basic supply of invariant submanifolds of the KdV equation. This is due to the property of the vector fields of the hierarchy to commute among themselves. From this property it follows that the set of singular point of any linear combination (with constant coefficients) of the vector fields of the hierarchy is a finite–dimensional invariant submanifold of the KdV flow. These submanifold can be usefully exploited to construct classes of solutions of this equation.

As a first example of this technique we consider the submanifold defined by the condition

$$\frac{\partial u}{\partial t_3} = \lambda \frac{\partial u}{\partial t_1} \,, \tag{1.22}$$

that is, the submanifold where the second vector field of the hierarchy is a constant multiple of the first one. It is formed by the solutions of the third order differential equation

$$\frac{1}{4}(u_{xxx} - 6uu_x) - \lambda u_x = 0 \;. \tag{1.23}$$

Therefore it is a three dimensional manifold, which we denote by M_3. We can use as coordinates on M_3 the *values* of the function u and its derivatives u_x and u_{xx} at *any* point x_0. To avoid cumbersome notations, we will continue to denote these three *numbers* with the same symbols, u, u_x, u_{xx}, but the reader should be aware of this subtlety. To perform the reduction of the first equation of the hierarchy (1.21) on M_3, we consider the first two differential consequences of the equation $\frac{\partial u}{\partial t_1} = u_x$ and we use the constraint (1.22) to eliminate the third order derivative. We obtain the system

$$\frac{\partial u}{\partial t_1} = u_x, \quad \frac{\partial u_x}{\partial t_1} = u_{xx}, \quad \frac{\partial u_{xx}}{\partial t_1} = 6uu_x + 4\lambda u_x \;. \tag{1.24}$$

We call X_1 the vector field defined by these equations on M_3. It shares many of the properties of the KdV equation. For instance, it is related to a Poisson pair. The simplest way to display this property is to remark that X_1 possesses two integrals of motion,

$$\begin{aligned} H_0 &= u_{xx} - 3u^2 - 4\lambda u \\ H_1 &= -\frac{1}{2}u_x^2 + u^3 + 2\lambda u^2 + uH_0 \;. \end{aligned} \tag{1.25}$$

Then we notice that on M_3 there exists a unique Poisson bracket $\{\cdot,\cdot\}_0$ with the following two properties:

i) The function H_0 is a Casimir function, that is, $\{F, H_0\}_0 = 0$ for every smooth function F on M_3.

ii) X_1 is the Hamiltonian vector field associated with the function H_1.

Such a Poisson bracket $\{\cdot,\cdot\}_0$ is defined by the relations

$$\{u, u_x\}_0 = -1, \quad \{u, u_{xx}\}_0 = 0, \quad \{u_x, u_{xx}\}_0 = 6u + 4\lambda \ . \tag{1.26}$$

Similarly, one can notice that on M_3 there exists a unique Poisson bracket $\{\cdot,\cdot\}_1$ with the following "dual" properties:

i') The function H_1 is a Casimir function, that is, $\{F, H_1\}_1 = 0$ for every smooth function F on M_3.

ii') X_1 is the Hamiltonian vector field associated with the function H_0.

This second Poisson bracket $\{\cdot,\cdot\}_1$ is defined by the relations

$$\{u, u_x\}_1 = u, \quad \{u, u_{xx}\}_1 = u_x, \quad \{u_x, u_{xx}\}_1 = u_{xx} - u(6u + 4\lambda) \ . \tag{1.27}$$

Exercise 1.3 Verify the stated properties of the Poisson pair (1.26) and (1.27). □

We now exploit the previous remarks to understand the geometry of the flow associated with X_1. First we use the Hamiltonian representation

$$\frac{dF}{dt} = X_1(F) = \{F, H_1\}_0 \ . \tag{1.28}$$

It entails that the level surfaces of the Casimir function H_0 are two–dimensional (symplectic) manifolds to which X_1 is tangent. Let us pick up any of these symplectic leaves, for instance the one passing through the origin $u = 0, u_x = 0, u_{xx} = 0$. Let us call S_2 this leaf. The coordinates (u, u_x) are *canonical* coordinates on S_2. The level curves of the Hamiltonian H_1 define a Lagrangian foliation of S_2. Our problem is to find the flow of the vector field X_1 along these Lagrangian submanifold. We have already given the solution of this problem in the particular case of the Lagrangian submanifold passing through the origin. This submanifold is the one–dimensional invariant submanifold (1.13) previously discussed in connection with Darboux transformations. The relative flow is the one–soliton solution to KdV. To deal with the generic Lagrangian submanifolds on an equal footing, it is useful to change strategy and to use the Hamilton–Jacobi equation

$$H_1(u, \frac{dW}{du}) = E \ . \tag{1.29}$$

In this rather simple example, there is almost nothing to say about this equation (which is obviously solvable), and the second Poisson bracket (1.26) seems not to play any role in the theory.

This (wrong!) impression is promptly corrected by the study of a more elaborated example. Let us consider the five–dimensional submanifold M_5 of

the singular points of the third vector field of the KdV hierarchy. It is defined by the equation

$$u_{xxxxx} - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x = 0 \ . \tag{1.30}$$

On this submanifold we can consider the restrictions of the first *two* vector fields of the same hierarchy. To compute the reduced equation we proceed as before. We regard the Cauchy data $(u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$ as coordinates on M_5. then we compute the time derivatives $\dfrac{\partial u}{\partial t_1}, \dfrac{\partial u_x}{\partial t_1}, \dfrac{\partial u_{xx}}{\partial t_1}, \dfrac{\partial u_{xxx}}{\partial t_1}, \dfrac{\partial u_{xxxx}}{\partial t_1}$ by taking the differential consequences of $\dfrac{\partial u}{\partial t_1} = u_x$, and by using (1.30) and its differential consequences as a constraint to eliminate all the derivatives of degree higher than four. We obtain the equations

$$\begin{aligned}
\frac{\partial u}{\partial t_1} &= u_x \\
\frac{\partial u_x}{\partial t_1} &= u_{xx} \\
\frac{\partial u_{xx}}{\partial t_1} &= u_{xxx} \\
\frac{\partial u_{xxx}}{\partial t_1} &= u_{xxxx} \\
\frac{\partial u_{xxxx}}{\partial t_1} &= 10uu_{xxx} + 20u_x u_{xx} - 30u^2 u_x
\end{aligned} \tag{1.31}$$

In the same way, for the reduction of the KdV equation, we get

$$\begin{aligned}
\frac{\partial u}{\partial t_3} &= \frac{1}{4}(u_{xxx} - 6uu_x) \\
\frac{\partial u_x}{\partial t_3} &= \frac{1}{4}(u_{xxxx} - 6uu_{xx} - 6u_x^2 \\
\frac{\partial u_{xx}}{\partial t_3} &= \frac{1}{4}(4uu_{xxxx} + 2u_x u_{xx} - 30u^2 u_x) \\
\frac{\partial u_{xxx}}{\partial t_3} &= \frac{1}{4}(2u_{xx}^2 + 6u_x u_{xxx} + 4uu_{xxxx} - 60uu_x^2 - 30u^2 u_{xx}) \\
\frac{\partial u_{xxxx}}{\partial t_3} &= \frac{1}{4}(10u_{xx}u_{xxx} + 10u_x u_{xxxx} - 120u^3 u_x - 100uu_x u_{xx} \\
&\quad + 10u^2 u_{xxx} - 60u_x^3) \ .
\end{aligned} \tag{1.32}$$

Exercise 1.4 Verify the previous computations. □

To find the corresponding solutions of the KdV equation, regarded as a partial differential equation in x and t, we have to:

1. Construct a common solution to the ordinary differential equations (1.31) and (1.32);

2. Consider the first component $u(t_1, t_3)$ of such a solution;

3. Set $t_1 = x$ and $t_3 = t$.

The function $u(x,t)$ obtained in this way is the solution we were looking for. What makes this procedure worth of interest is that the ODEs (1.31)–(1.32) can be solved by means of a variety of methods. In particular, they can be solved by means of the method of separation of variables[4]. It can be shown that they are rather special equations: They are Hamiltonian with respect to a Poisson pair; this Poisson pair allows to foliate the manifold M_5 into four–dimensional symplectic leaves with special properties; each symplectic leaf S_4 carries a Lagrangian foliation to which the vector fields (1.31) and (1.32) are tangent; the Poisson pair defines a special set of coordinates on each S_4; in these coordinates the Hamilton–Jacobi equations associated with the Hamiltonian equations (1.31) and (1.32) can be simultaneously solved by additive separation of variables. Most of these properties will be proved in the next lecture.

1.4 The modified KdV hierarchy

We leave for the moment the theme of the reduction, and come back to the KdV hierarchy in its general form. We notice that the first equations (1.21) can also be written in the form

$$\begin{aligned}\frac{\partial u}{\partial t_1} &= (Q_1 - \lambda Q_0)\alpha_0\\ \frac{\partial u}{\partial t_3} &= (Q_1 - \lambda Q_0)(\lambda\alpha_0 + \alpha_1)\\ \frac{\partial u}{\partial t_5} &= (Q_1 - \lambda Q_0)(\lambda^2\alpha_0 + \lambda\alpha_1 + \alpha_2)\end{aligned} \tag{1.33}$$

This representation shows that these equations are Hamiltonian with respect to the whole Poisson pencil. This elementary property can be exploited to simply construct the modified KdV hierarchy. Let us write in general

$$\frac{\partial u}{\partial t_{2j+1}} = (Q_1 + \lambda Q_0)\alpha^{(j)}(\lambda)\ , \tag{1.34}$$

where

$$\alpha^{(j)}(\lambda) = \lambda^j\alpha_0 + \lambda^{j-1}\alpha_1 + \cdots + \alpha_j\ . \tag{1.35}$$

By means of the Miura map (1.6) we pull–back the one–forms $\alpha^{(j)}$ to one–form $\beta^{(j)}$ defined on the phase space of the modified equation according to the transformation law of one-forms,

$$\beta(h) = \Phi_h'^{\,*}\alpha(\Phi(h))\ . \tag{1.36}$$

[4]The fact that the stationary reductions of KdV can be solved by separation of variables is well-known (see, e.g., [9]). This classical method has recently found a lot of interesting new applications, as shown in the survey [26].

We then define the equations

$$\frac{\partial h}{\partial t_{2j+1}} = P_h \beta^{(j)}(\lambda) \ . \tag{1.37}$$

They are Φ–related to the corresponding equations of the KdV hierarchy, exactly as the mKdV *equation* is Φ–related to the KdV *equation*. Indeed,

$$\frac{\partial u}{\partial t_{2j+1}} = \Phi'_h \frac{\partial h}{\partial t_{2j+1}} = \Phi'_h {\Phi'_h}^* \alpha^{(j)}(\Phi(h);\lambda) = Q_u \alpha^{(j)}(u;\lambda) \ . \tag{1.38}$$

It is therefore natural to call equations (1.37) the *modified KdV hierarchy*. By using the explicit form of the operators P_h and ${\Phi'_h}^*$, it is easy to check that the modified hierarchy is defined by the conservation laws

$$\frac{\partial h}{\partial t_{2j+1}} = \partial_x H^{(2j+1)}, \tag{1.39}$$

where

$$H^{(2j+1)} = -\frac{1}{2}\alpha_x^{(j)} + \alpha^{(j)} h \ . \tag{1.40}$$

Exercise 1.5 Write down explicitly the first three equations of the modified hierarchy. □

The above formulas are basic in the Sato approach. In the fourth lecture, after a more accurate analysis of the Hamiltonian structure of the KdV equations, we shall be led to consider the currents $H^{(2j+1)}$ as defining a point of an infinite–dimensional Grassmannian. This point evolves in time as the point u moves according to the KdV equation. We shall determine the equation of motion of the currents $H^{(2j+1)}$. They define a "bigger" hierarchy called the *Central System*. This system contains the KdV hierarchy as a particular reduction. It enjoys the property of being linearizable. In this way, by a continuous process of extension motivated by the Hamiltonian structure of the equations (from the single KdV equation to the KdV hierarchy and to the Central System), we arrive at the result that the KdV flow can be linearized. At this point the following picture of the possible strategies for solving the KdV equations emerges: either we pass to the Sato infinite–dimensional Grassmannian and we use a linearization technique, or we restrict the equation to a finite–dimensional invariant submanifold and we use a technique of separation of variables. The two strategies complement themselves rather well. Our attitude is to see the Grassmannian picture as a compact way of defining the equations, and the "reductionist" picture as an effective way for finding interesting classes of solutions.

The plan of the lectures

This is the web of ideas which we would like to make more precise in the following lectures. As cornerstone of our presentation we choose the concept of Poisson pairs. In the second lecture, we develop the theory of these pairs up to the point of giving a sound basis to the concept of Lenard chain. In the third lecture we exhibit a first class of examples, and we explain a reduction technique allowing to construct the Poisson pairs of the reduced flows. In the fourth lecture we give a second look at the KdV theory, and we explain the reasons which, according to the Hamiltonian standpoint, suggest to pass on the infinite–dimensional Sato Grassmannian. In the fifth lecture we better explore the relation between the two strategies, and we touch the concept of Lax representation. Finally, the last lecture is devoted to the method of separation of variables. The purpose is to show how the geometry of the reduced Poisson pairs can be used to define the separation coordinates.

2 The method of Poisson pairs

In the previous lecture we have outlined the birth of the method of Poisson pairs and its main purpose: To define integrable hierarchies of vector fields. In this lecture we dwell on the theoretical basis of this construction presenting the concept of Gel'fand–Zakharevich system.

The starting point is the notion of *Poisson manifold*. A manifold is said to be a Poisson manifold[5] if a composition law on scalar functions has been defined obeying the usual properties of a Poisson bracket: bilinearity, skewsymmetry, Jacobi identity and Leibnitz rule. The last condition means that that the Poisson bracket is a derivation in each entry:

$$\{fg, h\} = \{f, h\}g + f\{g, h\} . \tag{2.1}$$

Therefore, by fixing the argument of one of the two entries and keeping free the remaining one, we obtain a vector field,

$$X_h = \{\cdot, h\} . \tag{2.2}$$

It is called the *Hamiltonian vector field* associated with the function h with respect to the given Poisson bracket. Due to the remaining conditions on the Poisson bracket, these vector fields are closed with respect to the commutator. They form a Lie algebra homeomorphic to the algebra of functions defined by the Poisson bracket:

$$[X_f, X_g] = X_{\{f,g\}} . \tag{2.3}$$

Therefore a Poisson bracket on a manifold has a twofold role: it defines a Lie algebra structure on the ring of C^∞–functions, and provides a representation of this algebra on the manifold by means of the Hamiltonian vector fields.

Instead of working with the Poisson bracket, it is often suitable to work (especially in the infinite–dimensional case) with the associated Poisson tensor. It is the bivector field P on M defined by

$$\{f, g\} = \langle df,\, P\, dg\rangle . \tag{2.4}$$

In local coordinates, its components $P^{jk}(x^1, \dots, x^n)$ are the Poisson brackets of the coordinate functions,

$$P^{jk}(x^1, \dots, x^n) = \{x^j, x^k\} . \tag{2.5}$$

By looking at this bivector field as a linear skewsymmetric map $P : T^*M \to TM$, we can define the Hamiltonian vector fields X_f as the images through P of the exact one-forms,

$$X_f = P\, df . \tag{2.6}$$

In local coordinates this means

$$X_f^j = P^{jk}\frac{\partial f}{\partial x^k} . \tag{2.7}$$

[5]The books [17] and [31] are very good references for this topic.

Exercise 2.1 Show that the components of the Poisson tensor satisfy the cyclic condition

$$\sum_l \left(P^{jl}\frac{\partial P^{km}}{\partial x^l} + P^{kl}\frac{\partial P^{mj}}{\partial x^l} + P^{ml}\frac{\partial P^{jk}}{\partial x^l} \right) = 0 \ . \tag{2.8}$$

□

Exercise 2.2 Suppose that M is an affine space A. Call V the vector space associated with A. Define a bivector field on A as a mapping $P : A \times V^* \to V$ which satisfies the skewsymmetry condition

$$\langle \alpha, \, P_u\beta \rangle = -\langle \beta, \, P_u\alpha \rangle \tag{2.9}$$

for every pair of covector (α, β) in V^* and at each point $u \in A$. Denote the directional derivative at the point u of the mapping $u \mapsto P_u\alpha$ along the vector v by

$$P'_u(\alpha; v) = \frac{d}{dt} P_{u+tv}\alpha|_{t=0} \ . \tag{2.10}$$

Show that the bivector P is a Poisson bivector if and only if it satisfies the cyclic condition

$$\langle \alpha, \, P'_u(\beta; P_u\gamma) \rangle + \langle \beta, \, P'_u(\gamma; P_u\alpha) \rangle + \langle \gamma, \, P'_u(\alpha; P_u\beta) \rangle = 0 \ . \tag{2.11}$$

□

Exercise 2.3 Check that the bivector Q_λ of equation (1.17), associated with the KdV equation, fulfills the conditions (2.9) and (2.11). □

No condition is usually imposed on the rank of the Poisson bracket, that is, on the dimension of the vector space spanned by the Hamiltonian vector fields at each point of the manifold. If these vector fields span the whole tangent space the bracket is said to be regular, and the manifold M turns out to be a symplectic manifold. Indeed there exists, in this case, a unique symplectic 2-form ω such that

$$\{f, g\} = \omega(X_f, X_g) \ . \tag{2.12}$$

More interesting is the case where the bracket is singular. In this case, the Hamiltonian vector fields span a proper distribution D on M. It is involutive but, generically, not of constant rank. Nonetheless, this distribution is *completely integrable*: at each point there exists an integral submanifold of maximal dimension which is tangent to the distribution. These submanifolds are symplectic manifolds, and are called the symplectic leaves of the Poisson structure. The symplectic form is still defined by equation (2.12). Indeed, even if there

is not a 1–1 correspondence between (differentials of) functions and Hamiltonian vector fields, this formula keeps its meaning, since the value of the Poisson bracket does not depend on the particular choice of the Hamiltonian function associated with a given Hamiltonian vector field. We arrive thus at the following conclusion: a Poisson manifold is either a symplectic manifold or a *stratification* of symplectic manifolds possibly of different dimensions. It can be proven that, in a sufficiently small open set where the rank of the Poisson tensor is constant, these symplectic manifolds are the level sets of some smooth functions $F_1, \dots, F_k$, whose differentials span the kernel of the Poisson tensor. They are called *Casimir functions* of P (see below).

Exercise 2.4 Let $\{x_1, x_2, x_3\}$ be Cartesian coordinates in $M = \mathbb{R}^3$. Prove that the assignment

$$\{x_1, x_2\} = x_3, \quad \{x_1, x_3\} = -x_2, \quad \{x_2, x_3\} = x_1 \tag{2.13}$$

defines a Poisson tensor on M. Find its Casimir function, and describe the symplectic foliation associated with it. □

After these brief preliminaries on Poisson manifolds as natural settings for the theory of Hamiltonian vector fields, we pass to the theory of *bi-Hamiltonian* manifolds. Our purpose is to provide evidence that they are a suitable setting for the theory of *integrable* Hamiltonian vector fields. The simplest connection between the theory of integrable Hamiltonian vector fields and the theory of bi-Hamiltonian manifold is given by the Gel'fand–Zakharevich (GZ) theorem [13, 14] we shall discuss in this lecture.

A bi-Hamiltonian manifold is a Poisson manifold endowed with a *pair* of *compatible* Poisson brackets. We shall denote these brackets with $\{f, g\}_0$ and $\{f, g\}_1$. They are compatible if the Poisson pencil

$$\{f, g\}_\lambda := \{f, g\}_1 - \lambda\{f, g\}_0 \tag{2.14}$$

verifies the Jacobi identity for any value of the continuous (say, real) parameter λ. By means of this concept we catch the main features of the situation first met in the KdV example of Lecture 1. The new feature deserving attention is the dependence of the Poisson bracket (2.14) on the parameter λ. It influences all the objects so far introduced on a Poisson manifold: Hamiltonian fields and symplectic foliation. In particular, this foliation changes with λ. The useful idea is to extract from this moving foliation the invariant part. It is defined as the intersection of the symplectic leaves of the pencil when λ ranges over $\mathbb{R} \cup \{\infty\}$. The GZ theorem describes the structure of these intersections in particular cases.

Let us suppose that the dimension of M is odd, $\dim M = 2n + 1$, and that the rank of the Poisson pencil is maximal. This means that the dimension of the characteristic distribution spanned by the Hamiltonian vector field is $2n$ for almost all the values of the parameter λ, and almost everywhere on the manifold M. In this situation the generic symplectic leaf of the pencil has accordingly dimension $2n$ and the intersection of all the symplectic leaves are submanifolds

of dimension n. For brevity, we shall call this intersection the *support* of the pencil. The GZ theorem displays an important property of the leaves of the support of the pencil.

Theorem 2.5 *On a $(2n+1)$–dimensional bi-Hamiltonian manifold, whose Poisson pencil has maximal rank, the leaves of the support are generically Lagrangian submanifolds of dimension n contained on each symplectic leaf of dimension $2n$.*

This theorem contains two different statements. First of all it states that the dimension of the support is exactly half of the dimension of the generic symplectic leaf. It is the "hard" part of the theorem. Then it claims that the leaves of the support are Lagrangian submanifolds. Contrary to the appearances, this is the easiest part of the theorem, as we shall see. To better understand the content of the GZ theorem, we deem suitable to look at it from a different and, so to say, more constructive, point of view. It requires the use of the concept of *Casimir function*, defined as a function which commutes with all the other functions with respect to the Poisson bracket. Equivalently, it can be defined as a function whose differential belongs to the kernel of the Poisson tensor, i.e., a function generating the null vector field. In the case of a Poisson pencil, the Casimir functions depend on the parameter λ. If the rank of the Poisson pencil is maximal, the Casimir function is essentially unique (two Casimir functions are functionally dependent). The main content of the GZ theorem is that there exists a Casimir function depending polynomially on the parameter λ, and that the degree of this polynomial is exactly n if $\dim M = 2n+1$. Thus we can write the Casimir function in the form

$$C(\lambda) = C_0\lambda^n + C_1\lambda^{n-1} + \cdots + C_n \ . \qquad (2.15)$$

This result means that the Poisson pencil selects $n+1$ distinguished functions $(C_0, C_1, \ldots C_n)$. Generically these functions are independent. Their common level surfaces are the leaves of the support of the pencil. Indeed, on the support the function $C(\lambda)$ must be constant independently of the particular value of λ. Thus all the coefficients $(C_0, C_1, \ldots C_n)$ must be separately constant. Furthermore, as a consequence of the fact that $C(\lambda)$ is a Casimir function, it is easily seen that the coefficients C_k verify the Lenard recursion relations,

$$\{\cdot, C_k\}_1 = \{\cdot, C_{k+1}\}_0 \ , \qquad (2.16)$$

together with the vanishing conditions

$$\{\cdot, C_0\}_0 = \{\cdot, C_n\}_1 = 0 \ . \qquad (2.17)$$

In the language of the previous lecture, the functions $(C_0, C_1, \ldots C_n)$ form a *Lenard chain.* A typical property of these functions is to be mutually in involution:

$$\{C_j, C_k\}_0 = \{C_j, C_k\}_1 = 0 \ . \qquad (2.18)$$

This is proved by repeatedly using the recursion relation (2.16) to go back and forth along the chain. It follows that the leaves of the support are isotropic submanifolds, but since they are of maximal dimension n they are actually Lagrangian submanifolds. These short remarks should give a sufficiently detailed idea of the meaning of the GZ theorem.

Exercise 2.6 Check that that the integrals of motion H_0 and H_1 of the reduced flow X_1 on the invariant submanifold M_3 considered in Section 1.3 are the coefficients of the Casimir function $C(\lambda) = \lambda H_0 + H_1$ of the Poisson pencil defined on M_3. □

Exercise 2.7 Prove the claim (2.18) about the involutivity of the coefficients of a Casimir polynomial. □

From our standpoint, the above results are worthwhile of interest for two different reasons: First of all they show how the Lenard recursion relations, characteristic of the theory of "soliton equations", arise in a theoretically sound way in the framework of bi-Hamiltonian manifold. Secondly, they highlight the existence of a distinguished set of Hamiltonian $(C_0, C_1, \dots C_n)$ on the manifold M. Let us now choose one of the brackets of the pencil, say the bracket $\{\cdot,\cdot\}_0$. The function C_0 is a Casimir function for this bracket, and therefore its level surfaces are the synplectic leaves of the bracket $\{\cdot,\cdot\}_0$. Let us call ω_0 the symplectic 2–form defined on these submanifolds. As a consequence of the involution relation (2.18), the restrictions of the n functions $(C_1, \dots C_n)$ to the symplectic leaf are in involution with respect to ω_0. According to the Arnol'd–Liouville theorem, they define a family (or "hierarchy") of n completely integrable Hamiltonian vector fields on the symplectic leaf.

Definition 2.8 *The family of completely integrable Hamiltonian systems defined by the functions $(C_1, \dots C_n)$ on each symplectic leaf of the Poisson bracket $\{\cdot,\cdot\}_0$ will be called the GZ hierarchy associated with the Poisson pencil $\{\cdot,\cdot\}_\lambda$ defined on the bi-Hamiltonian manifold M.*

We shall be particularly interested in the study of this hierarchy for two reasons. First we want to show that the previous simple concepts allow to reconstruct a great deal of the KdV hierarchy, up to the linearization process on the infinite–dimensional Sato Grassmannian. In other words, we want to show that the theory of Poisson pairs is a natural gate to the theory of infinite-dimensional Hamiltonian systems described by partial differential equations of evolutionary type. Secondly, in a finite-dimensional setting, we want to show that the GZ vector fields are often more than integrable in the Liouville sense. Indeed, under some mild additional assumptions on the Poisson pencil, they are separable, and the separation coordinates are dictated by the geometry of the bi-Hamiltonian manifold. This result strenghtens the connection between Poisson pairs and integrability.

3 A first class of examples and the reduction technique

The aim of this lecture is to present a first class of nontrivial examples of GZ hierarchies. The examples are constructed to reproduce the reduced KdV flows discussed in the first lecture. The relation, however, will not be immediately manifest, and the reader has to wait until the fifth lecture for a full understanding of the motivations for some particular choice herewith made.

This lecture is split into three parts. In the first one we introduce a simple class of bi-Hamiltonian manifolds called Lie–Poisson manifolds. They are duals of Lie algebras endowed with a special Poisson pencil of Lie-theoretical origin. The Hamiltonian vector fields defined on these manifolds admit a *Lax representation* with a Lax matrix depending linearly on the parameter λ. In the second part we show how to combine several copies of these manifolds, in such a way to obtain Hamiltonian vector fields admitting a Lax representation depending polynomially on the parameter λ. Finally, in the third part, we introduce the geometrical technique of reduction of Marsden and Ratiu. It will allow us to specialize the form of the Lax matrix. The contact with the KdV theory, to be done in the fifth lecture, will then consist in showing that the reduced KdV flows admit exactly the Lax representation of the Hamiltonian vector fields considered in this lecture. This will ascertain the bi-Hamiltonian character of the reduced KdV flows. The lecture ends with an example worked out in detail.

3.1 Lie–Poisson manifolds

In this section $M = \mathfrak{g}^*$ is the dual of a Lie algebra $\mathfrak{g}$. We denote by S a point in M, and by $\dfrac{\partial F}{\partial S}$ the differential of a function $F : M \to \mathbb{R}$. This differential is the unique element of the algebra $\mathfrak{g}$ such that

$$\frac{dF}{dt} = \left\langle \frac{\partial F}{\partial S}, \dot{S} \right\rangle \tag{3.1}$$

along any curve passing through the point S. The Poisson pencil on M is defined by

$$\{F, G\}_\lambda = \left\langle S + \lambda\, A, \left[\frac{\partial F}{\partial S}, \frac{\partial G}{\partial S}\right] \right\rangle , \tag{3.2}$$

where A is any fixed element in $\mathfrak{g}^*$. In all the examples related to the KdV theory, $\mathfrak{g} = \mathfrak{sl}(2)$, S and $\dfrac{\partial F}{\partial S}$ are traceless 2×2 matrices, and

$$A = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} . \tag{3.3}$$

The Hamiltonian vector field X_F has the form

$$\dot{S} = \left[S + \lambda\, A, \frac{\partial F}{\partial S} \right] \ . \tag{3.4}$$

It is already in Lax form, with a Lax matrix given by

$$L(\lambda) = \lambda\, A + S \ . \tag{3.5}$$

Exercise 3.1 Compute the Poisson tensor and the Hamiltonian vector fields associated with the pencil (3.2). □

3.2 Polynomial extensions

We consider two copies of the algebra $\mathfrak{g}$. Accordingly, we denote by (S_0, S_1) a point in M and by $\left(\frac{\partial F}{\partial S_0}, \frac{\partial F}{\partial S_1} \right)$ the differential of the function $F : M \to \mathbb{R}$. By definition, along any curve $t \mapsto (S_0(t), S_1(t))$ we have

$$\frac{d}{dt} F = \left\langle \frac{\partial F}{\partial S_0}, \dot{S}_0 \right\rangle + \left\langle \frac{\partial F}{\partial S_1}, \dot{S}_1 \right\rangle \ . \tag{3.6}$$

The two copies of the algebra are intertwined by the Poisson brackets. As a Poisson pair on M we choose the following brackets

$$\begin{aligned} \{F, G\}_0 &= \left\langle A, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_1} \right] + \left[\frac{\partial F}{\partial S_1}, \frac{\partial G}{\partial S_0} \right] \right\rangle + \left\langle S_1, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_0} \right] \right\rangle \\ \{F, G\}_1 &= \left\langle A, \left[\frac{\partial F}{\partial S_1}, \frac{\partial G}{\partial S_1} \right] \right\rangle - \left\langle S_0, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_0} \right] \right\rangle \end{aligned} \tag{3.7}$$

The motivations can be found for instance in [20] (see also [25]). Later on we shall see how to extend this definition to the case of an arbitrary number of copies.

Exercise 3.2 Check that equations (3.7) indeed define a Poisson pair. □

Let us now study the Hamiltonian vector fields. Those defined by the brackets $\{\cdot, \cdot\}_0$ have the form

$$\begin{aligned} \dot{S}_0 &= \left[A, \frac{\partial F}{\partial S_1} \right] + \left[S_1, \frac{\partial F}{\partial S_0} \right] \\ \dot{S}_1 &= \left[A, \frac{\partial F}{\partial S_0} \right] \ . \end{aligned} \tag{3.8}$$

Those defined by the second bracket $\{\cdot, \cdot\}_1$ are

$$\begin{aligned} \dot{S}_0 &= - \left[S_0, \frac{\partial F}{\partial S_0} \right] \\ \dot{S}_1 &= \left[A, \frac{\partial F}{\partial S_1} \right] \ . \end{aligned} \tag{3.9}$$

It turns out that the Hamiltonian vector fields associated with the Poisson pencil are given by

$$
\begin{aligned}
\dot{S}_0 &= -\left[S_0 + \lambda\, S_1, \frac{\partial F}{\partial S_0}\right] - \left[\lambda\, A, \frac{\partial F}{\partial S_1}\right] \\
\dot{S}_1 &= -\left[\lambda\, A, \frac{\partial F}{\partial S_0}\right] + \left[A, \frac{\partial F}{\partial S_1}\right] \ .
\end{aligned}
\tag{3.10}
$$

This computation allows to display an interesting property of these vector fields. If we multiply the second equation by λ and add the result to the first equation we find

$$
(\lambda^2\, A + \lambda\, S_1 + S_0)^\bullet = \left[\frac{\partial F}{\partial S_0}, \lambda^2\, A + \lambda\, S_1 + S_0\right] \ . \tag{3.11}
$$

This is a Lax representation with Lax matrix $L(\lambda) = \lambda^2\, A + \lambda\, S_1 + S_0$. It depends polynomially on the parameter of the pencil. We have thus ascertained that all the Hamiltonian vector fields relative to the Poisson pencil (3.10) admit a Lax representation. The converse, however, is not necessarily true. Indeed, it must be noticed that the single Lax equation (3.11) is not sufficient to completely reconstruct the Poisson pencil (3.10). Additional constraints on the matrix $L(\lambda)$ are required to make the problem well-posed. The kind of constraints to be set are suggested by the geometric theory of reduction which we shall now outline.

3.3 Geometric reduction

We herewith outline a specific variant [5] of the reduction technique of Marsden and Ratiu [21] for Poisson manifolds. This variant is particularly suitable for bi-Hamiltonian manifolds.

Among the geometric objects defined by a Poisson pair (P_0, P_1) on a manifold M we consider:

i) a symplectic leaf S of one of the two Poisson bivectors, say P_0.

ii) the annihilator $(TS)^0$ of the tangent bundle of S, spanned by the 1–forms vanishing on the tangent spaces to S.

iii) the image $D = P_1(TS)^0$ of this annihilator according to the second Poisson bivector P_1. It is spanned by the Hamiltonian vector fields associated with the Casimir functions of P_0 by P_1.

iv) the intersection $E = D \cap TS$ of the distribution D with the tangent bundle of the selected symplectic leaf S.

It can be show that E is an integrable distribution as a consequence of the compatibility of the Poisson brackets. Therefor we can consider the space of leaves of this distribution, $N = S/E$. We assume N to be a smooth manifold. By the Marsden–Ratiu theorem, N is a reduced bi-Hamiltonian manifold.

The reduced brackets on N can be computed by using the process of "prolongation of functions" from N to M. Given any function $f : N \to \mathbb{R}$, we consider it as a function on S, invariant along the leaves of E. Then we choose any function $F : M \to \mathbb{R}$ which annihilates D and coincides with f on S. This function is said to be a prolongation of f. It is not unique, but this fact is not disturbing. It can be show that, if F and G are prolongations of f and g, their bracket $\{F, G\}_\lambda$ is an invariant function along E. Therefore it defines a function on N which is by definition the reduced bracket $\{f, g\}_\lambda$. The final result, of course, is independent of the particular choices of the prolongations F and G.

3.4 An explicit example

According to the spirit of these lectures, rather than discussing the proof of the reduction theorem stated in Section 3.3, we prefer to illustrate it on a concrete example. Let us thus perform the reduction of the Poisson pencil defined on two copies of $\mathfrak{g} = \mathfrak{sl}(2)$. The matrices S_0 and S_1 are traceless matrices whose entries we denote as follows:

$$S_0 = \begin{pmatrix} p_0 & r_0 \\ q_0 & -p_0 \end{pmatrix}, \qquad S_1 = \begin{pmatrix} p_1 & r_1 \\ q_1 & -p_1 \end{pmatrix}. \tag{3.12}$$

The space M has dimension six, and the entries of S_0 and S_1 are global coordinates on it. In these coordinates the differential of a function $F : M \to \mathbb{R}$ is represented by the pair of matrices

$$\frac{\partial F}{\partial S_0} = \begin{pmatrix} \frac{1}{2}\frac{\partial F}{\partial p_0} & \frac{\partial F}{\partial q_0} \\ \frac{\partial F}{\partial r_0} & -\frac{1}{2}\frac{\partial F}{\partial p_0} \end{pmatrix}, \qquad \frac{\partial F}{\partial S_1} = \begin{pmatrix} \frac{1}{2}\frac{\partial F}{\partial p_1} & \frac{\partial F}{\partial q_1} \\ \frac{\partial F}{\partial r_1} & -\frac{1}{2}\frac{\partial F}{\partial p_1} \end{pmatrix}. \tag{3.13}$$

Exercise 3.3 Define the pairing $\left\langle \frac{\partial F}{\partial S}, \dot{S} \right\rangle$ on $\mathfrak{g}$ as the trace of the product of the matrices $\frac{\partial F}{\partial S}$ and $\dot{S}$. Show that the matrices (3.13) verify the defining equation (3.6). $\square$

The Hamiltonian vector fields (3.8) and (3.9) are consequently given by

$$\begin{aligned}
\dot{p}_0 &= r_1\frac{\partial F}{\partial r_0} - q_1\frac{\partial F}{\partial q_0} - \frac{\partial F}{\partial q_1}\\
\dot{q}_0 &= q_1\frac{\partial F}{\partial p_0} - 2p_1\frac{\partial F}{\partial r_0} + \frac{\partial F}{\partial p_1}\\
\dot{r}_0 &= 2p_1\frac{\partial F}{\partial q_0} - r_1\frac{\partial F}{\partial p_0}\\
\dot{p}_1 &= -\frac{\partial F}{\partial q_0}\\
\dot{q}_1 &= \frac{\partial F}{\partial p_0}\\
\dot{r}_1 &= 0
\end{aligned} \tag{3.14}$$

and by

$$\begin{aligned}
\dot{p}_0 &= -r_0\frac{\partial F}{\partial r_0} + q_0\frac{\partial F}{\partial q_0}\\
\dot{q}_0 &= 2p_0\frac{\partial F}{\partial r_0} - q_0\frac{\partial F}{\partial p_0}\\
\dot{r}_0 &= -2p_0\frac{\partial F}{\partial q_0} + r_0\frac{\partial F}{\partial p_0}\\
\dot{p}_1 &= -\frac{\partial F}{\partial q_1}\\
\dot{q}_1 &= \frac{\partial F}{\partial p_1}\\
\dot{r}_1 &= 0
\end{aligned} \tag{3.15}$$

respectively.

Step 1: The reduced space N.

First we notice that the Hamiltonian vector fields (3.14) verify the constraints

$$\dot{r}_1 = 0\ , \qquad (r_0 + p_1^2 + r_1 q_1)^\bullet = 0\ . \tag{3.16}$$

It follows that the submanifold $S \subset M$ defined by the equations

$$r_1 = 1\ , \qquad r_0 + p_1^2 + r_1 q_1 = 0 \tag{3.17}$$

is a symplectic leaf of the first Poisson bivector P_0. Furthermore, it follows that the annihilator $(TS)^0$ is spanned by the exact 1–forms dr_1 and $d(r_0+p_1^2+r_1q_1)$. By computing the images of these forms according to the second Poisson bivector (3.15), we find the distribution D. It is spanned by the single vector

field

$$
\begin{aligned}
\dot{p_0} &= -r_0 \\
\dot{q_0} &= 2p_0 \\
\dot{r_0} &= 0 \\
\dot{p_1} &= -1 \\
\dot{q_1} &= 2p_1 \\
\dot{r_1} &= 0
\end{aligned}
\tag{3.18}
$$

which verifies the five constraints

$$
\begin{aligned}
&(p_0 - r_0 p_1)^\bullet = 0, \qquad (q_0 + 2p_0 p_1 - r_0 p_1^2)^\bullet = 0, \\
&(q_1 + p_1^2)^\bullet = 0, \quad \dot{r_0} = 0, \quad \dot{r_1} = 0.
\end{aligned}
\tag{3.19}
$$

They show that $D \subset TS$, and therefore $E = D$. Moreover they yield that the leaves of E on S are the level curves of the functions

$$
\begin{aligned}
u_1 &= q_1 + p_1^2 \\
u_2 &= p_0 + p_1 q_1 + p_1^3 \\
u_3 &= q_0 + 2p_0 p_1 + q_1 p_1^2 + p_1^4
\end{aligned}
\tag{3.20}
$$

We conclude that:

- $N = \mathbb{R}^3$;
- (u_1, u_2, u_3) are global coordinates on N;
- the canonical projection $\pi : S \to S/E$ is defined by equations (3.20).

Step 2: The reduced brackets.

Consider any function $f : N \to \mathbb{R}$. The function

$$
F := f(q_1 + p_1^2, p_0 + p_1 q_1 + p_1^3, q_0 + 2p_0 p_1 + q_1 p_1^2 + p_1^4) \tag{3.21}
$$

is clearly a prolongation of f to M, since it coincides with f on S, and is invariant along D. We can thus use F to compute the first component of the reduced Hamiltonian vector field on N according to the following algorithm:

$$
\begin{aligned}
\dot{u_1} &\overset{(3.20)}{=} \dot{q_1} + 2p_1 \dot{p_1} \overset{(3.14)}{=} \frac{\partial F}{\partial p_0} - p_1 \frac{\partial F}{\partial q_0} \\
&\overset{(3.21)}{=} \left(\frac{\partial f}{\partial u_2} + 2p_1 \frac{\partial f}{\partial u_3} \right) - 2p_1 \frac{\partial f}{\partial u_3} = \frac{\partial f}{\partial u_2} \; .
\end{aligned}
\tag{3.22}
$$

The other components are evaluated in the same way. The final result is that the Hamiltonian vector fields associated with the reduced Poisson pencil on N

are defined by

$$
\begin{aligned}
\dot{u}_1 &= (u_1+\lambda)\frac{\partial f}{\partial u_2} + 2u_2\frac{\partial f}{\partial u_3} \\
\dot{u}_2 &= -(u_1+\lambda)\frac{\partial f}{\partial u_1} + (u_3 - 2\lambda u_1)\frac{\partial f}{\partial u_3} \\
\dot{u}_3 &= -2u_2\frac{\partial f}{\partial u_1} + (2\lambda u_1 - u_3)\frac{\partial f}{\partial u_2}
\end{aligned}
\tag{3.23}
$$

With this reduction process we passed from a six–dimensional manifold M to a three dimensional manifold N. Later on, we shall see that this manifold coincides with the invariant submanifold M_3 of KdV, defined by the constraint

$$
u_{xxx} - 6uu_x = 0\ . \tag{3.24}
$$

Step 3: the GZ hierarchy.

To compute the Casimir function of the pencil (3.23) we notice that these vector fields obey the constraint

$$
(2\lambda u_1 - u_3)\dot{u}_1 + 2u_2\dot{u}_2 - (u_1+\lambda)\dot{u}_3 = 0\ . \tag{3.25}
$$

Therefore, integrating this equation, we obtain that

$$
C(\lambda) = \lambda(u_1^2 - u_3) + (u_2^2 - u_1u_3) = \lambda C_0 + C_1 \tag{3.26}
$$

is the Casimir sought for. It fulfills the scheme of the GZ theorem, and it defines a "short" Lenard chain

$$
P_0 dC_0 = 0 \qquad P_1 dC_0 = P_0 dC_1 = X_1 \qquad P_1 dC_1 = 0\ . \tag{3.27}
$$

Therefore the GZ "hierarchy" consists of the single vector field

$$
X_1\ :\qquad
\begin{aligned}
\dot{u}_1 &= 2u_2 \\
\dot{u}_2 &= u_3 + 2u_1^2 \\
\dot{u}_3 &= 4u_1u_2
\end{aligned}
\tag{3.28}
$$

As a last remark, we notice that this vector field coincides with the restriction of the first equation $\dfrac{\partial u}{\partial t_1} = u_x$ of the KdV hierarchy on the invariant submanifold (3.24). Indeed, by the procedure explained in Section 1, the reduced equation written in the "Cauchy data coordinates" (u, u_x, u_{xx}) is given by

$$
\begin{aligned}
\frac{\partial u}{\partial t_1} &= u_x \\
\frac{\partial u_x}{\partial t_1} &= u_{xx} \\
\frac{\partial u_{xx}}{\partial t_1} &= 6uu_x
\end{aligned}
\tag{3.29}
$$

We can now pass from (3.29) to (3.28) by the change of variables

$$u_1 = \frac{1}{2}u, \quad u_2 = \frac{1}{4}u_x, \quad u_3 = \frac{1}{4}u_{xx} - \frac{1}{2}u^2\ . \tag{3.30}$$

This remark shows that the simplest reduced KdV flow is bi-Hamiltonian. In the fifth lecture we shall see that this property is general, and we shall explain the origin of the seemingly "ad hoc" change of variables (3.30).

3.5 A more general example

To deal with higher–order reduced KdV flows, we have to extend the class of bi-Hamiltonian manifolds to be considered. We outline the case of three copies of the algebra $\mathfrak{g}$. The formulas are similar to the ones of equation (3.7), albeit a little more involved. The brackets $\{F, G\}_0$ and $\{F, G\}_1$ are now given by

$$\begin{aligned}\{F, G\}_0 = &\left\langle A, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_2}\right] + \left[\frac{\partial F}{\partial S_1}, \frac{\partial G}{\partial S_1}\right] + \left[\frac{\partial F}{\partial S_2}, \frac{\partial G}{\partial S_0}\right]\right\rangle \\ &+ \left\langle S_2, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_1}\right] + \left[\frac{\partial F}{\partial S_1}, \frac{\partial G}{\partial S_0}\right]\right\rangle \\ &+ \left\langle S_1, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_0}\right]\right\rangle\end{aligned} \tag{3.31}$$

and

$$\begin{aligned}\{F, G\}_1 = &\left\langle A, \left[\frac{\partial F}{\partial S_1}, \frac{\partial G}{\partial S_2}\right] + \left[\frac{\partial F}{\partial S_2}, \frac{\partial G}{\partial S_1}\right]\right\rangle \\ &+ \left\langle S_2, \left[\frac{\partial F}{\partial S_1}, \frac{\partial G}{\partial S_1}\right]\right\rangle \\ &- \left\langle S_0, \left[\frac{\partial F}{\partial S_0}, \frac{\partial G}{\partial S_0}\right]\right\rangle.\end{aligned} \tag{3.32}$$

The comparison of the two examples allows to infer by induction the general rule for the Poisson pair, holding in the case of an arbitrary (finite) number of copies of $\mathfrak{g}$. The pencil (3.31)–(3.32) can be reduced according to the procedure shown before. If $\mathfrak{g} = \mathfrak{sl}(2)$ and A is still given by (3.3), the final result of the process is the following: We start from a nine–dimensional manifold M and, after reduction, we arrive at a five–dimensional manifold N. It fulfills the assumption of the GZ theorem. The GZ hierarchy consists of two vector fields, which are the reduced KdV flows given by (1.31) and (1.32).

Exercise 3.4 Perform the reduction of the pencil (3.31)–(3.32)) for $\mathfrak{g} = \mathfrak{sl}(2)$. $\square$

4 The KdV theory revisited

In this lecture we consider again the KdV theory, but from a new point of view. Our purpose is twofold. The first aim is to show that the KdV hierarchy is another example of GZ hierarchy. The second aim is to explain in which sense the KdV hierarchy can be linearized. The algebraic linearization procedure dealt with in this lecture was suggested for the first time by Sato [27] (see also the developments contained in [7, 28, 29]), who exploited the so–called Lax representation of the KdV hierarchy in the algebra of pseudo–differential operators. Here we shall give a different description, strictly related to the Hamiltonian representation of the KdV hierarchy as a kind of infinite-dimensional GZ hierarchy. However, the presentation does not go beyond the limits of a simple sketch of the theory. We refer to [10] for full details.

4.1 Poisson pairs on a loop algebra

In this section we consider the infinite-dimensional Lie algebra M of C^∞–maps from the circle S^1 into $\mathfrak{g} = \mathfrak{sl}(2)$. A generic point of this manifold is presently a 2×2 traceless matrix

$$S = \begin{pmatrix} p(x) & r(x) \\ q(x) & -p(x) \end{pmatrix} , \tag{4.1}$$

whose entries are periodic functions of the coordinate x running over the circle. The three functions (p, q, r) play the role of "coordinates" on our manifold. The scalar-valued functions $F : M \to \mathbb{R}$ to be considered are local functionals

$$F = \int_{S^1} f(p, q, r; p_x, q_x, r_x; \dots)\, dx \ . \tag{4.2}$$

As before, their differentials are given by the matrices

$$\frac{\delta F}{\delta S} = \begin{pmatrix} \frac{1}{2}\frac{\delta f}{\delta p} & \frac{\delta f}{\delta q} \\ \frac{\delta f}{\delta r} & -\frac{1}{2}\frac{\delta f}{\delta p} \end{pmatrix} , \tag{4.3}$$

whose entries are the variational derivatives of the Lagrangian density f with respect to the functions (p, q, r). The Poisson pencil is similar to the first one considered in the previous lecture (see equation (3.2)). It is defined by

$$\{F, G\}_\lambda = \left\langle S + \lambda A, \left[\frac{\delta F}{\delta S}, \frac{\delta G}{\delta S}\right] \right\rangle + \omega\left(\frac{\delta F}{\delta S}, \frac{\delta G}{\delta S}\right) . \tag{4.4}$$

It differs from the previous example by the addition of the nontrivial cocycle

$$\omega\,(a, b) = \int_{S^1} \frac{da}{dx}\, b\, dx \ . \tag{4.5}$$

This term is essential to generate partial differential equations. It is responsible for the appearance of the partial derivative in the expansion of the Hamiltonian vector fields

$$\dot{S} = \left(\frac{\delta F}{\delta S}\right)_x + \left[S + \lambda\, A, \frac{\delta F}{\delta S}\right] \ . \tag{4.6}$$

Exercise 4.1 Recall that a two–cocycle on $\mathfrak{g}$ is a bilinear skewsymmetric map $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{R}$ which verifies the cyclic condition

$$\omega(a, [b, c]) + \omega(b, [c, a]) + \omega(c, [a, b]) = 0 \ .$$

Using this identity and the periodic boundary conditions check that equation (4.6) defines a Poisson bivector. □

4.2 Poisson reduction

We apply the same reduction technique used in the previous lecture, avoiding to give all the details of the computations. They can be either worked out by exercise or found in [5, 19]

The first Poisson bivector P_0 is defined by

$$\dot{S} = \left[A, \frac{\delta F}{\delta S}\right] , \tag{4.7}$$

where A is still defined by equation (3.3). These Hamiltonian vector fields obey the only constraint $\dot{r} = 0$. Therefore the submanifold $\mathcal{S}$ formed by the matrices

$$S = \begin{pmatrix} p & 1 \\ q & -p \end{pmatrix} \tag{4.8}$$

is a symplectic leaf of P_0. The annihilator $(T\mathcal{S})^0$ is spanned by the differentials of the functionals $F : M \to \mathbb{R}$ depending only on the coordinate function r. Consequently, the distribution D is spanned by the vector fields

$$\begin{aligned} \dot{p} &= \frac{\delta f}{\delta r} \\ \dot{q} &= \left(\frac{\delta f}{\delta r}\right)_x - 2p\frac{\delta f}{\delta r} \\ \dot{r} &= 0 \end{aligned} \tag{4.9}$$

The distribution D is thus tangent to $\mathcal{S}$ and E coincides with D. The vector field (4.9) verifies the constraint

$$\dot{q} + 2p\dot{p} + \dot{p_x} = (q + p^2 + p_x)^\bullet = 0 \ . \tag{4.10}$$

It follows that the leaves of the distribution E are the level sets of the function

$$u = q + p^2 + p_x \ . \tag{4.11}$$

Therefore the quotient space N is the space of scalar functions $u : S^1 \to \mathbb{R}$, and (4.11) is the canonical projection $\pi : \mathcal{S} \to \mathcal{S}/E$. We see that the manifold N is (isomorphic to) the phase space of the KdV equation.

We use the projection (4.11) to compute the reduced Poisson bivectors. The scheme of the computation is always the same. First we prolong any functional $\mathcal{F} = \int_{S^1} f(u, u_x, \cdots)dx$ on N into the functional

$$F(p, q, r) = \int_{S^1} f(q + p^2 + p_x, q_x + 2pp_x + p_{xx}; \dots)\, dx \tag{4.12}$$

on $\mathcal{S}$. Then we compute its differential at the points of $\mathcal{S}$,

$$\frac{\delta F}{\delta S} = \begin{pmatrix} -\frac{1}{2}\left(\frac{\delta F}{\delta u}\right)_x + p\frac{\delta F}{\delta u} & \frac{\delta F}{\delta u} \\ 0 & \frac{1}{2}\left(\frac{\delta F}{\delta u}\right)_x - p\frac{\delta F}{\delta u} \end{pmatrix} . \tag{4.13}$$

Finally, we evaluate the reduced Hamiltonian vector fields on N according to the usual scheme:

$$\begin{aligned}
\dot{u} &\overset{(4.11)}{=} \dot{q} + \dot{p_x} + 2p\dot{p} \\
&\overset{(4.6)}{=} \left[\left(\frac{\delta f}{\delta u}\right)_x + (q+\lambda)\frac{\delta f}{\delta p} - 2p\frac{\delta f}{\delta r}\right] + \left[\frac{1}{2}\left(\frac{\delta f}{\delta p}\right)_x + \frac{\delta f}{\delta r} + (q+\lambda)\frac{\delta f}{\delta q}\right]_x \\
&\quad + 2p\left[\frac{1}{2}\left(\frac{\delta f}{\delta p}\right)_x + \frac{\delta f}{\delta r} + (q+\lambda)\frac{\delta f}{\delta q}\right] \\
&\overset{(4.13)}{=} (q+\lambda)\left[-\left(\frac{\delta f}{\delta u}\right)_x + 2p\frac{\delta f}{\delta u}\right] + \left[-\frac{1}{2}\left(\frac{\delta f}{\delta u}\right)_{xx} + \left(p\frac{\delta f}{\delta u}\right)_x + (q+\lambda)\frac{\delta f}{\delta u}\right]_x \\
&\quad 2p\left[-\frac{1}{2}\left(\frac{\delta f}{\delta u}\right)_{xx} + \left(p\frac{\delta f}{\delta u}\right)_x + (q+\lambda)\frac{\delta f}{\delta u}\right] \\
&= -\frac{1}{2}\left(\frac{\delta f}{\delta u}\right)_{xxx} + 2(q + p_x + p^2 + \lambda)\left(\frac{\delta f}{\delta u}\right)_x + (q_x + p_{xx} + 2pp_x)\frac{\delta f}{\delta u} \\
&\overset{(4.11)}{=} -\frac{1}{2}\left(\frac{\delta f}{\delta u}\right)_{xxx} + 2(u+\lambda)\left(\frac{\delta f}{\delta u}\right)_x + u_x\frac{\delta f}{\delta u}.
\end{aligned} \tag{4.14}$$

We obtain the Poisson pencil of the KdV equation. This pencil is therefore the reduction of the "canonical" pencil (4.4) over a loop algebra.

4.3 The GZ hierarchy

The simplest way for computing the Casimir function of the above pencil is to use the Miura map. Since this map relates the pencil to the simple bivector

of the mKdV equation, it is sufficient to compute the Casimir of the latter bivector, and to transform it back to the phase space of the KdV equation.

We notice that the Casimir function of the mKdV hierarchy (1.39) is given by

$$\mathsf{H}(h) = 2\,z \int_{S^1} h\,dx \ , \tag{4.15}$$

where the constant z has been inserted for future convenience.

To obtain the Casimir function of the KdV equation, we must "invert" the Miura map by expressing h as a function of u. To do that we exploit the dependence of the Miura map on the parameter $\lambda = z^2$ of the pencil. We know that in the finite-dimensional case the Casimir function can be found as a polynomial in λ. In the infinite-dimensional case, we expect the Casimir function to be represented by a series. It is then natural to look at h in the form of a Laurent series in z,

$$h(z) = z + \sum_{l\geq 1} h_l z^{-l} \ , \tag{4.16}$$

whose coefficients h_l are scalar-valued periodic functions of x. In this way we change our point of view on the Miura map. Henceforth it must be looked at as a relation between a scalar function u and a Laurent series $h(z)$. This change of perspective deeply influences all the mKdV theory. It is a possible starting point for the Sato picture of the KdV theory, as we shall show later.

By inserting the expansion (4.16) into the Miura map $h_x + h^2 = u + z^2$ and equating the coefficients of different powers of z, we easily compute recursively the coefficients h_l as differential polynomial of the function u. The first ones are

$$\begin{aligned}
h_1 &= \tfrac{1}{2}u \\
h_2 &= -\tfrac{1}{4}u_x \\
h_3 &= \tfrac{1}{8}(u_{xx} - u^2) \\
h_4 &= -\tfrac{1}{16}(u_{xxx} - 4uu_x) \\
h_5 &= \tfrac{1}{32}(u_{xxxx} - 6uu_{xx} - 5u_x^2 + 2u^3) \ .
\end{aligned} \tag{4.17}$$

One can notice (see [1]) that all the even coefficients h_{2l} are total x–derivatives. This remark explains the "strange" enumeration with odd times used for the KdV hierarchy in the first lecture.

To compute concretely the GZ vector fields, besides the Casimir function

$$\mathsf{H}(u,z) = 2z \sum_{l\geq 1} \int_{S^1} h_l z^{-l}\,dx \ , \tag{4.18}$$

we need its differential. To simplify the notation we set

$$\alpha := \frac{\delta \mathsf{H}}{\delta u} = 1 + \sum_{l\geq 1} \alpha_l z^{-l} \ . \tag{4.19}$$

Once again, the simplest way for evaluating this series is to use the Miura map. We notice that $\beta = 2z$ is the differential of the Casimir of the mKdV equation. From the transformation law of 1-forms,

$$\Phi_h'^{\,*}(\alpha) = \beta \ , \tag{4.20}$$

we then conclude that that α solves the equation

$$-\alpha_x + 2\alpha h = 2z \ . \tag{4.21}$$

As before, the coefficients α_l can be computed recursively. One finds a Laurent series in $\lambda = z^2$,

$$\alpha = 1 - \frac{1}{2}u\lambda^{-1} + \frac{1}{8}(3u^2 - u_{xx})\lambda^{-2} + \cdots \ , \tag{4.22}$$

whose first coefficients have already appeared in (1.20). From α we can easily evaluate the Lenard partial sums $\alpha^{(j)} = \left(\lambda^j \alpha\right)_+$ and write the odd GZ equations in the form

$$\frac{\partial u}{\partial t_{2j+1}} = \Big(-\frac{1}{2}\partial_{xxx} + 2(u+\lambda)\partial_x + u_x \Big)\left(\alpha^{(j)}\right) \ . \tag{4.23}$$

The even ones are

$$\frac{\partial u}{\partial t_{2j}} = 0 \ . \tag{4.24}$$

The above equations completely and tersely define the KdV hierarchy from the standpoint of the method of Poisson pairs.

4.4 The Central System

We shall now pursue a little further the far–reaching consequences of the change of point of view introduced in the previous subsection. According to this new point of view, the mKdV hierarchy is defined in the space $\mathcal{L}$ of the Laurent series in z truncated form above. This affects the whole picture.

Let us consider again the basic formulas of the mKdV theory. They are the Miura map,

$$h_x + h^2 = u + z^2 \ , \tag{4.25}$$

the formula for the currents (1.40),

$$H^{(2j+1)} = -\frac{1}{2}\alpha_x^{(j)} + \alpha^{(j)}h, \quad H^{(2j)} = 0 \ , \tag{4.26}$$

and the definition of the mKdV hierarchy

$$\frac{\partial h}{\partial t_j} = \partial_x H^{(j)} \ . \tag{4.27}$$

They were obtained in the first lecture. Presently they are complemented by the information that $h(z)$ is a Laurent series of the form (4.16). We shall now investigate the meaning of the above formulas in this new setting.

We start form the series $h(z)$, and we associate with it a new family of Laurent series $h^{(j)}(z)$ defined recursively by

$$h^{(j+1)} = h^{(j)}_x + hh^{(j)} \ , \tag{4.28}$$

starting from $h^{(0)} = 1$. They form a *moving frame* associated with the point h in the space of (truncated) Laurent series. The first three elements of this frame are explicitly given by

$$h^{(0)} = 1, \qquad h^{(1)} = h, \qquad h^{(2)} = h_x + h^2 \ . \tag{4.29}$$

We see the basic block $h_x + h^2$ of the Miura transformation appearing. We call $\mathcal{H}_+$ the linear span of the series $\{h^{(j)}\}_{j\geq 0}$. It is a linear subspace of $\mathcal{L}$, attached to the point h. We can now interpret the three basic formulas of the mKdV theory as properties of this linear space:

- The Miura map (4.25) tells us that the linear space $\mathcal{H}_+$ is invariant with respect to the multiplication by λ,

$$\lambda(\mathcal{H}_+) \subset \mathcal{H}_+ \ . \tag{4.30}$$

- The formula (4.26) for the currents then entails that the currents $H^{(j)}$, for $j \in \mathbb{N}$, belong to $\mathcal{H}_+$:

$$H^{(j)} \in \mathcal{H}_+ \ . \tag{4.31}$$

- Furthermore, in conjunction with equation (4.21), it entails that the asymptotic expansion of the currents $H^{(j)}$ has the form

$$H^{(j)} = z^j + \sum_{l\geq 1} H^i_l z^{-l} = z^j + O(z^{-1}) \ . \tag{4.32}$$

- Finally, the mKdV equations (4.27) can be seen as the commutativity conditions of the operators $(\partial_x + h)$ and $(\frac{\partial}{\partial t_j} + H^{(j)})$:

$$\left[\partial_x + h, \frac{\partial}{\partial t_j} + H^{(j)}\right] = 0 \ . \tag{4.33}$$

Used together, conditions (4.31) and (4.33) imply that the operators $\left(\frac{\partial}{\partial t_j}+H^{(j)}\right)$ leave the linear space $\mathcal{H}_+$ invariant:

$$\left(\frac{\partial}{\partial t_j}+H^{(j)}\right)(\mathcal{H}_+)\subset\mathcal{H}_+ \ . \tag{4.34}$$

This is the abstract but simple form of the laws governing the time evolution of the currents $H^{(j)}$. These equations are the "top" of the KdV theory, and form the basis of the Sato theory. It is not difficult to give them a concrete form. By using the form of the expansion (4.32) it is easy to show that equations (4.34) are equivalent to the infinite system of Riccati–type equations on the currents $H^{(j)}$:

$$\frac{\partial H^{(j)}}{\partial t_k}+H^{(j)}H^{(k)}=H^{(j+k)}+\sum_{l=1}^{j}H_l^kH^{(j-l)}+\sum_{l=1}^{k}H_l^jH^{(k-l)} \ . \tag{4.35}$$

It will be called the Central System (CS).

Exercise 4.2 Prove formulas (4.31) and (4.32).

4.5 The linearization process

The first reward of the previous work is the discovery of a linearization process. The equations (4.35) of the Central System are not directly linearizable, but they can be easily transformed into a new system of linearizable Riccati equations by a transformation in the space of currents. This idea is realized once again by a "Miura map". The novelty, however, is that this map is now operating on the space of currents rather than on the phase space of the KdV equation.

We simply give the final result. Let us consider a new family of currents $\{W^{(k)}\}_{k\geq 0}$ of the form

$$W^{(k)}=z^k+\sum_{l\geq 1}W_l^kz^{-l} \ , \tag{4.36}$$

and let us denote by $\mathcal{W}_+$ their linear span in $\mathcal{L}$. We define (see also [29]) a new system of equations on the currents $W^{(k)}$ by imposing the "constraints"

$$\left(\frac{\partial}{\partial t_k}+z^k\right)(\mathcal{W}_+)\subset\mathcal{W}_+ \tag{4.37}$$

on their linear span $\mathcal{W}_+$. It is easily seen that they take the explicit form

$$\frac{\partial W^{(k)}}{\partial t_j}+z^jW^{(k)}=W^{(j+k)}+\sum_{l=1}^{j}W_l^kW^{(j-l)} \ . \tag{4.38}$$

They will be called the *Sato equations* (on the "big cell of the Sato Grassmannian"). They are a system of linearizable Riccati equations. This can be seen

either from the geometry of a suitable group action on the Grassmannian [7] or by means of the following more elementary considerations. We write equations (4.38) in the matrix form

$$\frac{\partial \mathsf{W}}{\partial t_j} + \mathsf{W} \cdot {}^T\Lambda^k - \Lambda^k \cdot \mathsf{W} = \mathsf{W}\Gamma_k\mathsf{W} \ , \tag{4.39}$$

where $\mathsf{W} = \left(W_l^k\right)$ is the matrix of the components of the currents $W^{(k)}$, Λ is the infinite shift matrix

$$\Lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & & \\ 0 & 0 & 1 & 0 & \cdots & \\ \vdots & & \ddots & \ddots & & \\ \vdots & & & \ddots & \ddots & \\ \vdots & & & & \ddots & \end{bmatrix} \ , \tag{4.40}$$

and Γ^k is the convolution matrix of level k,

$$\Gamma_k = \begin{bmatrix} 0 & \cdots & & 1 & 0 & \cdots \\ \vdots & & 1 & 0 & \cdots & \cdots \\ & . & . & & & \\ 1 & 0 & & & & \\ \vdots & & & & & \end{bmatrix} \ . \tag{4.41}$$

One can thus check that the matrix Riccati equation (4.39) is solved by the matrix

$$\mathsf{W} = \mathsf{V} \cdot \mathsf{U}^{-1} \ , \tag{4.42}$$

where U and V satisfy the constant coefficients linear system

$$\frac{\partial}{\partial t_k}\mathsf{U} = {}^T\Lambda^k\mathsf{U} - \Gamma_k\mathsf{V} \ , \qquad \frac{\partial}{\partial t_k}\mathsf{V} = \Lambda^k\mathsf{V} \ . \tag{4.43}$$

The closing remark is that the Sato equations are mapped into the Central System (4.35) by the following algebraic Miura map:

$$H^{(j)} = \frac{\sum_{l=0}^{j} W^0_{j-l} W^{(l)}}{W^{(0)}} \ . \tag{4.44}$$

The outcome of this long chain of extensions and transformations is the following algorithm for solving the KdV equation:

i) First we solve the linear system (4.43), with a suitably chosen initial condition, which we do not discuss here;

ii) Then we use the projective transformation (4.42) and the Miura map (4.44) to recover the currents $H^{(j)}$;

iii) Finally, we extract the first current $H^{(1)} = h$, and we evaluate the first component h_1 of its Laurent expansion in powers of z^{-1}.

The function

$$u(x, t_3, \ldots) = 2h_1|_{t_1=x} \tag{4.45}$$

is then a solution of the KdV equation.

4.6 The relation with the Sato approach

The equations (4.27) make sense for an arbitrary Laurent series h of the form (4.16), even if it is not a solution of the Riccati equation $h_x + h^2 = u + z^2$. Hence they define, for every j, a system of PDEs for the coefficients h_l. We will show[6] that these systems are equivalent to the celebrated KP hierarchy of the Kyoto school (see the lectures by Satsuma in these volume). The usual definition of the KP equations can be summarized as follows. Let $\Psi\mathcal{D}$ be the ring of pseudodifferential operators on the circle. It contains as a subring the space $\mathcal{D}$ of *purely differential* operators. Let us denote with $(\cdot)_+$ the natural projection from $\Psi\mathcal{D}$ onto $\mathcal{D}$. Let Q be a monic operators of degree 1,

$$Q = \partial - \sum_{j\geq 1} q_j \partial^{-j} \ . \tag{4.46}$$

The KP hierarchy is the set of Lax equations for Q

$$\frac{\partial}{\partial t_j} Q = \left[\left(Q^j \right)_+ , Q \right] \ . \tag{4.47}$$

The aim of this subsection is to show that such a Lax representation just arises as a kind of a Euler form of the equations (4.27). Before stating the next result, we must observe that the relations (4.28) can be solved backwards, in such a way to define the Faà di Bruno elements $h^{(j)}$ for all $j \in \mathbb{Z}$.

Proposition 4.3 *Suppose the series h of the form (4.16) to evolve according to a conservation law,*

$$\frac{\partial h}{\partial t} = \partial_x H, \tag{4.48}$$

[6] See also the papers [6, 32].

for an arbitrary H. Then the Faà di Bruno elements $h^{(j)}$, for $j \in \mathbb{Z}$, evolve according to

$$\left(\frac{\partial}{\partial t}+H\right)h^{(j)}=\sum_{k=0}^{\infty}\binom{j}{k}(\partial_x^k H)h^{(j-k)}\ , \tag{4.49}$$

where

$$\binom{j}{k}=\frac{j(j-1)\cdots(j-k+1)}{k!}\ ,\qquad \binom{j}{0}=1\ .$$

Now we consider the map $\phi : \mathcal{L} \to \Psi\mathcal{D}$, from the space of Laurent series to the ring of pseudodifferential operators on the circle, which acts on the Faà di Bruno basis according to

$$\phi(h^{(j)})=\partial^j. \tag{4.50}$$

This map is then extended by linearity (with respect to multiplication by a *function* of x) to the whole space $\mathcal{L}$.

Definition 4.4 *We call Lax operator of the KP theory the image*

$$Q=\phi(z) \tag{4.51}$$

of the first element of the standard basis in $\mathcal{L}$.

If the q_j are the components of the expansion of z on the Faà di Bruno basis,

$$z=h^{(1)}-\sum_{j\geq 1}q_j h^{(-j)}\ , \tag{4.52}$$

then we can write

$$Q=\partial-\sum_{j\geq 1}q_j\partial^{-j} \tag{4.53}$$

according to the definition of the map ϕ. We note that equation (4.52) uniquely defines the coefficients q_j as differential polynomials of the components h_j of $h(z)$:

$$\begin{aligned} &q_1=h_1,\quad q_2=h_2,\quad q_3=h_3+h_1^2\\ &q_4=h_4+3h_1h_2-h_1h_{1x}\\ &\cdots\cdots \end{aligned} \tag{4.54}$$

This is an invertible relation between the h_j and the q_j, so that equation (4.52) may be seen as a change of coordinates in the space $\mathcal{L}$.

Proposition 4.5 *The map ϕ has the following three properties:*

i) *Multiplying a vector of the Faà di Bruno basis by a power z^k of z yields*

$$\phi(z^k \cdot h^{(j)}) = \partial^j \cdot Q^k \ . \tag{4.55}$$

ii) *The evolution along a conservation law of the form*

$$\frac{\partial h}{\partial t} = \partial_x \left(\sum_k H_k z^k \right)$$

translates into

$$\frac{\partial}{\partial t} \left(\phi(h^{(j)})\right) = \sum_k [\partial^j, H_k] \cdot Q^k \ . \tag{4.56}$$

iii) *If π_+ and Π_+ are respectively the projection onto the positive part $\mathcal{H}_+ \subset \mathcal{L}$ and $\mathcal{D} \subset \Psi\mathcal{D}$, then*

$$\phi \circ \pi_+ = \Pi_+ \circ \phi \ . \tag{4.57}$$

To obtain the Sato form of the equations (4.27), we derive the equation

$$z = h^{(1)} - \sum_{l \geq 1} q_l h^{(-l)} \tag{4.58}$$

with respect to the time t_j, getting

$$\sum_{l \geq 1} \frac{\partial q_l}{\partial t_j} h^{(-l)} = \frac{\partial h^{(1)}}{\partial t_j} - \sum_{l \geq 1} q_l \frac{\partial h^{(-l)}}{\partial t_j} \ . \tag{4.59}$$

Applying the map ϕ to both sides of this equation we obtain

$$\sum_{l \geq 1} \frac{\partial q_l}{\partial t_j} \partial^{-l} = \sum_{k \geq 1} [\partial, H_k^j] Q^{-k} - \sum_{k \geq 1} q_l [\partial^{-l}, H_k^j] Q^{-k}, \tag{4.60}$$

or

$$\frac{\partial Q}{\partial t_j} + \sum_{k \geq 1} [Q, H_k^j] Q^{-k} = 0 \ . \tag{4.61}$$

Finally, we introduce the operator

$$B^{(j)} = \phi(H^{(j)}) = \phi \left(z^j + \sum_{k \geq 1} H_k^j z^{-k} \right) = Q^j + \sum_{k \geq 1} H_k^j Q^{-k} \tag{4.62}$$

associated with the current density $H^{(j)}$, and we note that

$$B^{(j)} = \phi(\pi_+(z^j))) = \left(\phi(z^j)\right)_+ = \left(Q^j\right)_+ \ . \tag{4.63}$$

Thus we can write (4.61) in the final form

$$\frac{\partial Q}{\partial t_j} + [Q, \left(Q^j\right)_+] = 0 \ , \tag{4.64}$$

which coincides with equation (4.47).

5 Lax representation of the reduced KdV flows

In this lecture we want to investigate more accurately the properties of the stationary KdV flows, that is, of the equations induced by the KdV hierarchy on the finite–dimensional invariant submanifolds of the singular points of any equation of the hierarchy. Examples of these reductions have already been discussed in the first lecture. In the third lecture we realized, in a couple of examples, that the reduced flows were still bi-Hamiltonian. Although not at all surprising, this property is somewhat mysterious, since it is not yet well understood how the Poisson pairs of the reductions are related to the original Poisson pairs of the KdV equation. Moreover, even if the subject is quite old and classical (see, e.g., [3, 9, 4]), it was still lacking in the literature a systematic and coordinate free proof that such reduced flows are bi-Hamiltonian (see, however, [2, 30]). In this lecture we will not provide such a proof, which is contained in [11], but we will give a sufficiently systematic algorithm to compute the reduced Poisson pair. This algorithm is based on the study of the Lax representation of the reduced equations.

5.1 Lax representation

In this section we associate a Lax matrix (polynomially depending on λ) with each element $H^{(j)}$. This matrix naturally arises from a change of basis in the linear space $\mathcal{H}_+$ attached to the point h. So far we have introduced two bases:

i) The moving frame $\{h^{(j)}\}$;

ii) The canonical basis $\{H^{(j)}\}$.

Presently we introduce a third basis by exploiting the constraint

$$\lambda(\mathcal{H}_+) \subset \mathcal{H}_+ \ , \tag{5.1}$$

characteristic of the KdV theory. The new basis is formed by the multiples $\{\lambda^j H^{(0)}, \lambda^j H^{(1)}\}$ of the first two currents. Formally we define

$$\text{iii) the Lax basis: } (\lambda^j, \lambda^j h) \ .$$

The use of this basis leads to a new representation of the currents $H^{(j)}$, where each current is written as a linear combination of the first two, $H^{(0)} = 1$ and $H^{(1)} = h$, with coefficients that are polynomials in λ. Let us consider a few examples:

$$\begin{aligned}
H^{(0)} &= 1 + 0 \cdot h \\
H^{(1)} &= 0 \cdot 1 + 1 \cdot h \\
H^{(2)} &= \lambda \cdot 1 + 0 \cdot h \\
H^{(3)} &= -h_2 \cdot 1 + (\lambda - h_1) \cdot h \\
H^{(4)} &= \lambda^2 \cdot 1 \\
H^{(5)} &= (-\lambda h_2 + h_1 h_2 - h_4) \cdot 1 + (\lambda^2 - \lambda h_1 + h_1^2 - h_3) \cdot h \ .
\end{aligned} \tag{5.2}$$

This new representation also affects our way of writing the action of the operators $\left(\dfrac{\partial}{\partial t_j} + H^{(j)}\right)$. Let these operators act on $H^{(0)}$ and $H^{(1)}$. For the basic invariance condition (4.34), we get an element in $\mathcal{H}_+$ which can be represented on the Lax basis. As a result we can write

$$\left(\frac{\partial}{\partial t_j} + H^{(j)}\right)\begin{bmatrix} 1 \\ h \end{bmatrix} = L^{(j)}(\lambda)\begin{bmatrix} 1 \\ h \end{bmatrix} , \tag{5.3}$$

where $L^{(j)}(\lambda)$ is the *Lax matrix* associated with the current $H^{(j)}$. We shall see below the explicit form of some of these matrices.

It becomes now very easy to rewrite the Central System in the form of equations on the Lax matrices $L^{(j)}(\lambda)$. We simply have to notice that the equations (4.35) entail the "exactness condition"

$$\frac{\partial H^{(j)}}{\partial t_k} = \frac{\partial H^{(k)}}{\partial t_j} , \tag{5.4}$$

from which it follows that the operators $\left(\dfrac{\partial}{\partial t_j} + H^{(j)}\right)$ and $\left(\dfrac{\partial}{\partial t_k} + H^{(k)}\right)$ commute:

$$\left[\frac{\partial}{\partial t_j} + H^{(j)}, \frac{\partial}{\partial t_k} + H^{(k)}\right] = 0 \ . \tag{5.5}$$

It is now sufficient to evaluate this condition on $(H^{(0)}, H^{(1)})$ and to expand on the Lax basis to find the "zero curvature representation" of the KdV hierarchy:

$$\frac{\partial L^{(j)}}{\partial t_k} - \frac{\partial L^{(k)}}{\partial t_j} + \left[L^{(j)}, L^{(k)}\right] = 0 \ . \tag{5.6}$$

Suppose now that we are on the invariant submanifold formed by the singular points of the j–th member of the KdV hierarchy. On this submanifold

$$\frac{\partial L^{(k)}}{\partial t_j} = 0 \qquad \forall\, k, \tag{5.7}$$

and the zero curvature representation becomes the Lax representation

$$\frac{\partial L^{(j)}}{\partial t_k} = \left[L^{(k)}, L^{(j)}\right] \ . \tag{5.8}$$

We have thus shown that all the stationary reductions of the KdV hierarchy admit a Lax representation. As a matter of fact, this Lax representation coincides [11] with the Lax representation of the GZ systems on Lie–Poisson manifolds studied in Section 3. The latter are bi-Hamiltonian systems. Therefore, we end up stating that the stationary reductions of the KdV theory are bi-Hamiltonian, and we can construct the associated Poisson pairs. We shall now see a couple of examples.

5.2 First example

We study anew the simplest invariant submanifold of the KdV hierarchy, defined by the equation

$$u_{xxx} - 6uu_x = 0 \ . \tag{5.9}$$

In this example we consider the constraint from the point of view of the Central System. Since the constraint is the stationarity of the time t_3, we have to consider only the first three Lax matrices. As for the matrix $L^{(1)}$, the following computation,

$$\begin{aligned}
&\big(\frac{\partial}{\partial t_1} + H^{(1)}\big)1 = 0 \cdot 1 + 1 \cdot h \\
&\big(\frac{\partial}{\partial t_1} + H^{(1)}\big)H^{(1)} \overset{(4.35)}{=} H^{(2)} + 2h_1 \overset{(5.2)}{=} (\lambda + 2h_1)\cdot 1 + 0 \cdot h \ ,
\end{aligned} \tag{5.10}$$

shows that

$$L^{(1)} = \begin{pmatrix} 0 & 1 \\ \lambda + 2\,h_1 & 0 \end{pmatrix} \ . \tag{5.11}$$

Similarly, the computation

$$\begin{aligned}
\big(\frac{\partial}{\partial t_3} + H^{(3)}\big)1 &\overset{(5.2)}{=} - h_2 \cdot 1 + (\lambda - h_1)\cdot h \\
\big(\frac{\partial}{\partial t_3} + H^{(3)}\big)H^{(1)} &\overset{(4.35)}{=} H^{(4)} + h_1 H^{(2)} + h_2 H^{(1)} + h_3 + H_1^3 \\
&\overset{(5.2)}{=} (\lambda^2 + \lambda h_1 + 2h_3 - h_1^2)\cdot 1 + h_2 \cdot h
\end{aligned} \tag{5.12}$$

yields

$$L^{(3)} = \begin{pmatrix} -h_2 & \lambda - h_1 \\ \lambda^2 + h_1\lambda + 2h_3 - {h_1}^2 & h_2 \end{pmatrix} . \tag{5.13}$$

On the submanifold M_3 defined by equation (5.9) this matrix verifies the Lax equation

$$\frac{\partial L^{(3)}}{\partial t_1} = \left[L^{(1)}, L^{(3)}\right] \ . \tag{5.14}$$

This equation completely defines the time evolution of the first three components (h_1, h_2, h_3) of the current $H^{(1)} = h$. These components play the role of coordinates on M_3. We get

$$\begin{aligned}
\frac{\partial h_1}{\partial t_1} &= -2h_2 \\
\frac{\partial h_2}{\partial t_1} &= -2h_3 - {h_1}^2 \\
\frac{\partial h_3}{\partial t_1} &= -4h_1 h_2
\end{aligned} \tag{5.15}$$

By the change of coordinates

$$h_1 = \frac{1}{2}u \ , \qquad h_2 = -\frac{1}{4}u_x \ , \qquad h_3 = \frac{1}{8}(u_{xx} - u^2) \ ,$$

coming from the inversion (4.15) of the Miura map, these equations take the form

$$\frac{\partial u}{\partial t_1} = u_x, \quad \frac{\partial u_x}{\partial t_1} = u_{xx}, \quad \frac{\partial u_{xx}}{\partial t_1} = 6uu_x \ , \tag{5.16}$$

already encountered in Lecture 1. This shows explicitly the connection between the two points of view.

To find the connection between these equations and the GZ equations dealt with in the first example of Lecture 3, we compare the Lax matrix

$$L^{(3)}(\lambda) = \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & 1 \\ h_1 & 0 \end{pmatrix} + \begin{pmatrix} -h_2 & -h_1 \\ 2h_3 - h_1^2 & h_2 \end{pmatrix}$$

with the Lax matrix

$$S(\lambda) = \lambda^2 \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \lambda \begin{pmatrix} p_1 & 1 \\ q_1 & -p_1 \end{pmatrix} + \begin{pmatrix} p_0 & -(q_1 + p_1^2) \\ q_0 & -p_0 \end{pmatrix}$$

associated with the points of the symplectic leaf defined by (3.17). We easily identify $L^{(3)}$ with the restriction of $S(\lambda)$ to $p_1 = 0$ upon setting

$$p_0 = -h_2, \quad q_1 = h_1, \quad q_0 = 2h_3 - h_1^2 \ . \tag{5.17}$$

By comparing these equations with the projection (3.20), which allows to pass from the symplectic leaf S to the quotient space $N = S/E$, we obtain the change of coordinates

$$u_1 = h_1, \quad u_2 = -h_2, \quad u_3 = 2h_3 - h_1^2, \tag{5.18}$$

connecting the reduction (5.15) of the Central System to the GZ system (3.28) dealt with in the third Lecture. The latter was, by construction, a bi-Hamiltonian system. We argue that also the reduction of the Central System herewith considered is a bi-Hamiltonian vector field, and that its Poisson pair is obtained by geometric reduction. Basic for this identification is the property of the Lax matrix $L^{(3)}$ of being a *section* of the fiber bundle $\pi : S \to S/E$ appearing in the geometric reduction. It is this property which allows to set an invertible relation among the coordinates (u_1, u_2, u_3), coming from the geometric reduction, and the coordinates (h_1, h_2, h_3) coming from the reduction of the Central System.

5.3 The generic stationary submanifold

It is now not hard to give the general form of the matrices $L^{(j)}$ for an arbitrary odd integer $2j+1$. First we observe that

$$(\frac{\partial}{\partial t_{2j+1}} + H^{(2j+1)})1 = H^{(2j+1)} \stackrel{(4.26)}{=} -\frac{1}{2}\alpha_x^{(j)} + \alpha^{(j)}h \ . \tag{5.19}$$

Then we notice that

$$\begin{aligned}(\frac{\partial}{\partial t_j} + H^{(j)})h &= H^{(j+1)} + \sum_{l=1}^{j} h_l H^{(j-l)} + H_1^j \\ &= -\frac{1}{2}\big(\alpha_x^{(j+1)} + \sum_{l=1}^{j} h_l \alpha_x^{(j-1)}\big) + H_1^j + \big(\alpha^{(j+1)} + \sum_{l=1}^{j} h_l \alpha^{(j-1)}\big)h \ .\end{aligned} \tag{5.20}$$

Therefore

$$L^{(j)} = \begin{pmatrix} -\frac{1}{2}\alpha_x^{(j)} & \alpha^{(j)} \\ -\frac{1}{2}(\alpha_x^{(j+1)} + \sum_{l=1}^{j} h_l \alpha_x^{(j-1)}) + H_1^j & \alpha^{(j+1)} + \sum_{l=1}^{j} h_l \alpha^{(j-1)} \end{pmatrix} . \tag{5.21}$$

By using the definition (4.21) of the Lenard series $\alpha(z)$ of which the polynomials $\alpha^{(j)}$ are the partial sums, it is easy to prove that $L^{(j)}$ is a traceless matrix.

We leave to the reader to specialize the matrix $L^{(5)}$, and to write explicitly the Lax equations

$$\frac{\partial L^{(5)}}{\partial t_1} = \big[L^{(1)}, L^{(5)}\big] \ , \qquad \frac{\partial L^{(5)}}{\partial t_3} = \big[L^{(3)}, L^{(5)}\big] \ . \tag{5.22}$$

They should be compared with the reduced KdV equations (1.31) and (1.32) on the invariant submanifold defined by the constraint

$$u_{xxxxx} - 10uu_{xxx} - 20u_x u_{xx} + 30u^2 u_x = 0 \ . \tag{5.23}$$

They should also be compared with the GZ equations obtained via the geometric reduction process applied to the Lie–Poisson pairs defined on three copies of $\mathfrak{sl}(2)$ by equations (3.31) and (3.32). We have not displayed explicitly these equations yet. We will give their form in the next lecture.

5.4 What more?

There is nothing "sacred" with the KdV theory. As we know, it is related with the constraint

$$z^2(\mathcal{H}_+) \subset (\mathcal{H}_+) \ , \tag{5.24}$$

which defines an invariant submanifold of the Central System. Many other constraints can be considered. For instance, the constraint

$$z^3(\mathcal{H}_+) \subset (\mathcal{H}_+) \tag{5.25}$$

leads to the so–called Boussinesq theory, and is studied in [12]. What is remarkable is that the change of constraint does not affect the algorithm for the study of the reduced equations. All the previous reasonings are valid without almost no change. The only difference resides in the fact that the computations become more involved. This remark allows to better appreciate the meaning of the process leading from the KdV equation to the Central System. We have not only given a new formulation to known equations. We have actually found a much bigger hierarchy, possessing remarkable properties, which coincides with the KdV hierarchy on a (small) proper invariant subset. The integrability properties belong to the bigger hierarchy, and hold outside the KdV submanifold. Many other interesting equations can be found by other processes of reduction. There is some evidence that a very large class of evolution equations possessing some integrability properties can be eventually recovered as a suitable reduction of the Central System, or of strictly related systems. However, we shall not pursue this point of view further, since it would lead us too far away from our next topic, the *separability* of the reduced KdV flows.

6 Darboux–Nijenhuis coordinates and Separability

In this lecture we shall consider the reduced KdV flows from a different point of view. Our aim is to probe the study of the geometry of the Poisson pair which, as realized in the third and fifth lectures, is associated with these flows. The final goal is to show the existence of a suitable set of coordinates defined by and adapted to the Poisson pair. They are called *Darboux–Nijenhuis* coordinates. We shall prove that they are separation coordinates for the Hamilton–Jacobi equations associated with the reduced flows.

To keep the presentation within a reasonable size, we shall mainly deal with a particular example, and we shall not discuss thoroughly the theoretical background, referring to [11] for more details. We shall use the example to display the characteristic features of the geometry of the reduced manifolds. The reader is asked to believe that all that will be shown is general inside the class of the reduced stationary KdV manifolds, whose Poisson pencils are of maximal rank. A certain care must be used in trying to extend these conclusions to other examples like the Boussinesq stationary reductions, whose Poisson pencils are not of maximal rank. They will not be covered in these lecture notes. The example worked out is the reduction of the first and the third KdV equations on the invariant submanifold defined by the equation

$$u_{xxxxx} - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x = 0 \ , \tag{6.1}$$

a problem addressed at the end of Section 5.3.

6.1 The Poisson pair

As we mentioned several times, the invariant submanifold M_5 defined by equation (6.1) has dimension five. From the standpoint of the Central System, it is characterized by the two equations

$$z^2(\mathcal{H}_+) \subset (\mathcal{H}_+) \ , \qquad H^{(5)}h = \lambda^3 + \sum_{l=1}^{5} h_l H^{(5-l)} + H_1^5 \ . \tag{6.2}$$

We recall that the first constraint means that, inside the big cell of the Sato Grassmannian, we are working on the special submanifold corresponding to the KdV theory. The second constraint means that, inside the phase space of the KdV theory, we are working on the set (6.1) of singular points of the fifth flow. The two constraints play the following roles. The first constraint sets up a relation among the currents $H^{(j)}$: All the currents are expressed as linear combinations (with polynomial coefficients) of the first two currents $H^{(0)} = 1$ and $H^{(1)} = h$. So this constraint drastically reduces the number of the unknowns H_l^j to the coefficients h_l of h. The second constraint then further cuts the degrees of freedom to a finite number, by setting relations among the coefficients h_l. It can be shown that only the first five coefficients $(h_1, h_2, h_3, h_4, h_5)$ survive as free

parameters. All the other coefficients can be expressed as polynomial functions of the previous ones. By a process of elimination of the exceeding coordinate, one proves that the restriction of the first and third flows of the KdV hierarchy are represented by the following differential equations:

$$\begin{aligned}
\frac{\partial h_1}{\partial t_1} &= -2h_2 \\
\frac{\partial h_2}{\partial t_1} &= -2h_3 - {h_1}^2 \\
\frac{\partial h_3}{\partial t_1} &= -2h_1h_2 - 2h_4 \\
\frac{\partial h_4}{\partial t_1} &= -2h_5 - {h_2}^2 - 2h_1h_3 \\
\frac{\partial h_5}{\partial t_1} &= -4h_3h_2 + 2{h_1}^2h_2 - 4h_1h_4
\end{aligned} \tag{6.3}$$

and

$$\begin{aligned}
\frac{\partial h_1}{\partial t_3} &= -2h_4 + 2h_1h_2 \\
\frac{\partial h_2}{\partial t_3} &= -2h_5 + {h_2}^2 + {h_1}^3 \\
\frac{\partial h_3}{\partial t_3} &= -2h_1h_4 + 4{h_1}^2h_2 - 2h_3h_2 \\
\frac{\partial h_4}{\partial t_3} &= -2{h_3}^2 - 2h_2h_4 + 2h_1{h_2}^2 + {h_1}^4 + {h_1}^2h_3 \\
\frac{\partial h_5}{\partial t_3} &= 2{h_1}^2h_4 - 4h_3h_4 + 2{h_1}^3h_2
\end{aligned} \tag{6.4}$$

They can also be seen as the Lax equations (5.22). However, for our purposes, it is more important to recognize that the above equations are the GZ equations of the Poisson pencil defined on M_5. This pencil can be computed according to the reduction procedure explained in the third lecture. The final outcome is

that the reduced Poisson bivector is given by

$$
\begin{aligned}
\dot{h}_1 &= 2\frac{\partial H}{\partial h_2} + 2(h_1 - \lambda)\frac{\partial H}{\partial h_4} + 2h_2\frac{\partial H}{\partial h_5} \\
\dot{h}_2 &= -2\frac{\partial H}{\partial h_1} + 2(\lambda - 2h_1)\frac{\partial H}{\partial h_3} - 2h_2\frac{\partial H}{\partial h_4} + (4\lambda h_1 - 2h_3 - h_1^2)\frac{\partial H}{\partial h_5} \\
\dot{h}_3 &= 2(2h_1 - \lambda)\frac{\partial H}{\partial h_2} + (2h_3 + 2h_1^2 - 4\lambda h_1)\frac{\partial H}{\partial h_4} + 2(h_4 + h_1 h_2)\frac{\partial H}{\partial h_5} \\
\dot{h}_4 &= 2(\lambda - h_1)\frac{\partial H}{\partial h_1} + 2h_2\frac{\partial H}{\partial h_2} - (2h_3 + 2h_1^2 - 4\lambda h_1)\frac{\partial H}{\partial h_3} \\
&\qquad + (2h_5 - 6h_1h_3 + h_2^2 + 2h_1^3 + 4\lambda h_3 + 2\lambda h_1^2)\frac{\partial H}{\partial h_5} \\
\dot{h}_5 &= -2h_2\frac{\partial H}{\partial h_1} + (2h_3 + h_1^2 - 4\lambda h_1)\frac{\partial H}{\partial h_2} - 2(h_4 + h_1h_2)\frac{\partial H}{\partial h_3} \\
&\qquad - (2h_5 - 6h_1h_3 + h_2^2 + 2h_1^3 + 4\lambda h_3 + 2\lambda h_1^2)\frac{\partial H}{\partial h_4} \, .
\end{aligned}
\tag{6.5}
$$

The Casimir function of this pencil is a quadratic polynomial,

$$
C(\lambda) = C_0\lambda^2 + C_1\lambda + C_2 \ , \tag{6.6}
$$

and the coefficients are

$$
\begin{aligned}
C_0 &= {h_1}^3 - 2h_1h_3 + h_5 \\
C_1 &= h_2h_4 - h_1h_5 + \tfrac{3}{2}{h_1}^2h_3 - \tfrac{1}{2}h_1{h_2}^2 - \tfrac{1}{2}{h_3}^2 - \tfrac{1}{2}{h_1}^4 \\
C_2 &= \tfrac{1}{2}h_3{h_2}^2 - h_3h_5 + \tfrac{1}{2}{h_1}^5 + h_1{h_3}^2 - h_1h_2h_4 - \tfrac{3}{2}{h_1}^3h_3 + {h_1}^2h_5 + \tfrac{1}{2}{h_4}^2
\end{aligned}
\tag{6.7}
$$

The Lenard chain is

$$
\begin{aligned}
P_0 dC_0 &= 0 \\
P_0 dC_1 &= P_1 dC_0 = \frac{\partial \mathbf{h}}{\partial t_1} \\
P_0 dC_2 &= P_1 dC_1 = \frac{\partial \mathbf{h}}{\partial t_3} \\
& \quad\; P_1 dC_2 = 0 \ ,
\end{aligned}
\tag{6.8}
$$

where $\mathbf{h}$ is the vector $(h_1, h_2, h_3, h_4, h_5)$. It shows that the reduced flows are bi-Hamiltonian. Finally, if one uses the coordinate change (4.17) from the coordinates $(h_1, h_2, h_3, h_4, h_5)$ to the coordinates $(u, u_x, u_{xx}, u_{xxx}, u_{xxxx})$, one can put the equations (6.3) and (6.4) in the form (1.31) and (1.32) considered in the first lecture.

6.2 Passing to a symplectic leaf

We aim to solve equations (6.3) and (6.4) by the Hamilton–Jacobi method. This requires to set the study of such equations on a symplectic manifold. This can be easily accomplished by noticing that these vector fields are already tangent to the submanifold S_4 defined by the equation

$$C_0 = E \ , \tag{6.9}$$

for a constant E. We know that this submanifold is symplectic since C_0 is the Casimir of P_0. The dimension of S_4 is four, and the variables (h_1, h_2, h_3, h_4) play the role of coordinates on it.

For our purposes it is crucial to remark an additional property of S_4: It is a bi-Hamiltonian manifold. This means that also the second bivector P_1 induces, by a process of reduction, a Poisson structure on S_4 compatible with the natural restriction of P_0. This is not a general situation. It holds as a consequence of a peculiarity of the Poisson pencil (6.5). The property we are mentioning concerns the vector field

$$Z = \frac{\partial}{\partial h_5}. \tag{6.10}$$

One can easily check that:

i) Z is transversal to the symplectic leaf S_4.

ii) The functions which are invariant along Z form a Poisson subalgebra with respect to the pencil.

In simpler terms, the Poisson bracket of functions which are independent of h_5 is independent on h_5 as well. Since they coincide with the functions on S_4 (by the transversality condition), this property allows us to define a pair of Poisson brackets also on S_4. The first bracket is associated with the symplectic 2–form ω_0 on S_4. It can be easily checked that

$$\omega_0 = h_1 dh_1 \wedge dh_2 + \frac{1}{2}(dh_2 \wedge dh_4 + dh_5 \wedge dh_1) \ . \tag{6.11}$$

The second Poisson bracket can be represented in the form

$$\{f, g\}_1 = \omega_0(NX_f, X_g) \ , \tag{6.12}$$

where X_f and X_g are the Hamiltonian vector fields associated with the functions f and g by the symplectic 2–form ω_0, and N is a $(1,1)$–tensor field on S_4, called the Nijenhuis tensor associated with the pencil (see, e.g., [15]). In our example one obtains

$$\begin{aligned} N =& \Big(- h_1 \frac{\partial}{\partial h_1} - h_2 \frac{\partial}{\partial h_2} + (h_3 - 3h_1^2) \frac{\partial}{\partial h_3} - 2h_1 h_2 \frac{\partial}{\partial h_4} \Big) \otimes dh_1 \\ &+ (h_3 - h_1^2) \frac{\partial}{\partial h_4} \otimes dh_2 + \Big(\frac{\partial}{\partial h_1} + 2h_1 \frac{\partial}{\partial h_3} + h_2 \frac{\partial}{\partial h_4} \Big) \otimes dh_3 \\ &+ \Big(\frac{\partial}{\partial h_2} + h_1 \frac{\partial}{\partial h_4} \Big) \otimes dh_4 \ . \end{aligned} \tag{6.13}$$

Thus we arrive at the following picture of the GZ hierarchy considered in this lecture. It is formed by a pair of vector fields, X_1 and X_3, defined by (6.3) and (6.4). They are tangent to the symplectic leaf (S_4, ω_0) defined by equations (6.9) and (6.11). This symplectic manifold is still bi-Hamiltonian, and therefore there exists a Nijenhuis tensor field N, defined by equation (6.12). The vector fields X_1 and X_3 span a Lagrangian subspace which is invariant with respect to N. One finds that they obey the following "modified Lenard recursion relations"

$$\begin{aligned} NX_1 &= X_3 + (\mathrm{Tr}\ N)X_1 \\ NX_3 &= \quad\ + (-\det\ N)X_1 \ . \end{aligned} \tag{6.14}$$

From them we can extract the matrix

$$\mathsf{F} = \begin{pmatrix} \mathrm{Tr}\ N & 1 \\ -\det\ N & 0 \end{pmatrix} \tag{6.15}$$

which represents the action of N on the abovementioned Lagrangian subspace. It will play a fundamental role in the upcoming discussion of the separability of the vector fields.

Exercise 6.1 Compute the expression of the reduced pencil on S_4 and check the form of the Nijenhuis tensor, as well as the modified Lenard recursion relations (6.14).

6.3 Darboux–Nijenhuis coordinates

We are now in a position to introduce the basic tool of the theory of separability in the bi-Hamiltonian framework: The concept of Darboux–Nijenhuis coordinates on a symplectic bi-Hamiltonian manifold, like S_4.

Given a symplectic 2–form ω_0 and a Nijenhuis tensor N coming from a Poisson pencil defined on a $2n$–dimensional manifold $\mathcal{M}$, under the assumption that the eigenvalues of N are real and functionally independent, one proves [18] the existence of a system of coordinates $(\lambda_1, \ldots, \lambda_n; \mu_1, \ldots, \mu_n)$ which are canonical for ω_0,

$$\omega_0 = \sum_{i=1}^{n} d\mu_1 \wedge d\lambda_i \ , \tag{6.16}$$

and which allows to put N^* (the adjoint of N) in diagonal form:

$$N^* d\lambda_i = \lambda_i d\lambda_i \ , \qquad N^* d\mu_i = \lambda_i d\mu_i \ . \tag{6.17}$$

The coordinates λ_i are the eigenvalues of N^*, and therefore can be computed as the *zeroes* of the minimal polynomial of N:

$$\lambda^n + c_1 \lambda^{n-1} + \cdots + c_n = 0 \ . \tag{6.18}$$

The coordinates μ_j can be computed as the *values* that a conjugate polynomial

$$\mu = f_1 \lambda^{n-1} + \cdots + f_n \tag{6.19}$$

assumes on the eigenvalues λ_j, that is,

$$\mu_j = f_1 \lambda_j^{n-1} + \cdots + f_n, \quad j = 1, \dots, n \ . \tag{6.20}$$

The determination of this polynomial, which is not uniquely defined by the geometric structures present in the theory, requires a certain care. Although there is presently a sufficiently developed theory on the Darboux–Nijenhuis coordinates and on their computation, for the sake of brevity we shall not tackle this problem, but rather limit ourselves to display these polynomials in the example at hand. They are

$$\begin{aligned} &\lambda^2 - h_1\lambda + (h_1^2 - h_3) = 0 \\ &\mu - h_2\lambda + (h_1 h_2 - h_4) = 0 \end{aligned} \tag{6.21}$$

The important idea emerging from the previous discussion is that the GZ equations are often coupled with a special system of coordinates related with the Poisson pair.

Exercise 6.2 Check that the polynomials (6.21) define a system of Darboux–Nijenhuis coordinates for the pair (ω_0, N) considered above.

6.4 Separation of Variables

We start from the classical Stäckel theorem on the separability, in orthogonal coordinates, of the Hamilton–Jacobi equation associated with the natural Hamiltonian

$$H(q,p) = \frac{1}{2} \sum g^{ii}(q) p_i^2 + V(q_1, \dots, q_n) \tag{6.22}$$

on the cotangent bundle of the configuration space. According to Stäckel, this Hamiltonian is separable if and only if there exists as invertible matrix $S(q_1, \dots, q_n)$ and a vector $U(q_1, \dots, q_n)$ such that H is among the solutions $(H_1, \dots, H_n)$ of the linear system

$$\frac{1}{2} p_i^2 = U_i(q) + \sum_{j=1}^{n} S_{ij}(q) H_j \ , \tag{6.23}$$

and S and U verify the Stäckel condition:

The rows of S and U depend only on the corresponding coordinate.

This means for instance that the elements S_{1j} and U_1 depend only on the first coordinate q_1, and so on. Such a matrix S is called a Stäckel matrix (and U a Stäckel vector).

The strategy we shall follow to prove the separability of the Hamilton–Jacobi equations associated with the GZ vector fields X_1 and X_3 on the manifold S_4 considered above, is to show that the Darboux–Nijenhuis coordinates allow to define a Stäckel matrix for the corresponding Hamiltonians.

The construction of the Stäckel matrix starts from the matrix F which relates the vector field X_1 and X_3 to the Nijenhuis tensor N (see equation (6.15)). One can prove that this matrix satisfies the remarkable identity

$$N^* d\mathsf{F} = \mathsf{F} d\mathsf{F} \ . \tag{6.24}$$

This is a matrix equation which must be interpreted as follows: $d\mathsf{F}$ is a matrix of 1–forms, and N^* acts separately on each entry of this matrix; $\mathsf{F}d\mathsf{F}$ denotes the matrix multiplication of the matrices F and $d\mathsf{F}$, which amounts to linearly combine the 1–forms appearing in $d\mathsf{F}$. In our example, equation (6.24) becomes

$$\begin{aligned} N^* d(\mathrm{Tr}N) &= -d(\det N) + (\mathrm{Tr}N)\, d(\mathrm{Tr}N) \\ N^* d(\det N) &= + (\det N)\, d(\mathrm{Tr}N) \end{aligned} \tag{6.25}$$

Exercise 6.3 Check that this equations are verified by the Nijenhuis tensor (6.13).

We leave for a moment the particular case we are dealing with, and we suppose that, on a symplectic bi-Hamiltonian manifold fulfilling the conditions of Subsection 6.3, a family of n vector fields $(X_1, X_3, \ldots, X_{2n-1})$ is given. We assume that they are Hamiltonian with respect to P_0, say, $X_{2i-1} = P_0 dC_i$, and that there exists a matrix F such that

$$N X_{2i-1} = \sum_{j=1}^{n} \mathsf{F}_i^j X_{2j-1} \qquad \text{for all } i \ . \tag{6.26}$$

Finally, we suppose that F satisfies condition (6.24). Then, from the matrix F we build up the matrix T whose rows are the left–eigenvectors of F. In other words, we construct a matrix T such that

$$\mathsf{F} = \mathsf{T}^{-1} \Lambda \mathsf{T} \ , \tag{6.27}$$

where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_n)$ is the diagonal matrix of the eigenvalues of F, coinciding with the eigenvalues of N. The matrix T is normalized by imposing that in each row there is a constant component. A suitable normalization criterion, for instance, is to set the entries in the last column equal to 1.

Theorem 6.4 *If the matrix F verifies condition (6.24) (as it is always true in our class of examples), then the matrix T is a (generalized) Stäckel matrix in the Darboux–Nijenhuis coordinates.*

This theorem means that the rows of the matrix T verify the following generalized Stäckel condition: The entries of the first row of T depend only on the

canonical pair (λ_1, μ_1), those of the second row on (λ_2, μ_2), and so on. With respect to the classical case recalled at the beginning of this lecture, we notice that by generalizing the class of Hamiltonians considered, we have been obliged to extend a little bit the notion of Stäckel matrix. However, this extension does not affect the theorem of separability. Indeed, as a consequence of the fact that the matrix F is defined by the vector fields $(X_1, X_3, \ldots, X_{2n-1})$ themselves through equation (6.26), one can prove that T is a Stäckel matrix for the corresponding Hamiltonians $(C_1, \ldots, C_n)$.

Theorem 6.5 *The column vector*

$$\mathbf{U} = \mathsf{T}\mathbf{C} \ , \tag{6.28}$$

where $\mathbf{C}$ is the column vector of the Hamiltonians $(C_1, \ldots, C_n)$, verifies the (generalized) Stäckel condition in the Darboux–Nijenhuis coordinates. This means that the first component of $\mathbf{U}$ depends only on the pair (λ_1, μ_1), the second on (λ_2, μ_2), and so on.

We shall not prove these two theorems here, preferring to see them "at work" in the example at hand. First we consider the matrix T. Due to the form (6.15) of the matrix F, it is easily proved that

$$\mathsf{T} = \begin{pmatrix} \lambda_1 & 1 \\ \lambda_2 & 1 \end{pmatrix} \ . \tag{6.29}$$

Indeed, the equation $\mathsf{TF} = \Lambda\mathsf{T}$ follows directly from the characteristic equation for the tensor N. It should be noted that the matrix T has been computed without computing explicitly the eigenvalues λ_1 and λ_2. It is enough to use the first of equations (6.21), defining the Darboux–Nijenhuis coordinates. The matrix T clearly possess the Stäckel property (even in the classical, restricted sense).

The vector $\mathbf{U}$ can be computed as well without computing explicitly the coordinates (λ_j, μ_j). It is sufficient, once again, to use the equations (6.21). We now pass to prove that equation (6.28), in our example, has the particular form

$$\begin{aligned} \frac{1}{2}\mu_1^2 - \frac{1}{2}\lambda_1^2 - E\lambda_1^2 &= \lambda_1 C_1 + C_2 \\ \frac{1}{2}\mu_2^2 - \frac{1}{2}\lambda_2^2 - E\lambda_2^2 &= \lambda_2 C_1 + C_2 \ . \end{aligned} \tag{6.30}$$

We notice that proving this statement is tantamount to proving that the following equality between polynomials,

$$\mu(\lambda)^2 - \lambda^5 = 2C(\lambda) \ , \tag{6.31}$$

is verified in correspondence of the eigenvalues of N. This can be done as follows. Let us write the polynomials defining the Darboux–Nijenhuis coordinates in the

symbolic form

$$\begin{aligned} \lambda^2 &= e_1\lambda + e_2 \\ \mu &= f_1\lambda + f_2 \ . \end{aligned} \tag{6.32}$$

The coefficients (e_j, f_j) of these polynomials must be regarded as known functions of the coordinates on the manifold. By squaring the second polynomial and by eliminating λ^2 by means of the first equation, we get

$$\mu^2 = f_1^2(e_1\lambda + e_2) + 2f_1f_2\lambda + f_1f_2 = (f_1^2e_1 + 2f_1f_2)\lambda + (f_1^2e_2 + f_2^2) \ . \tag{6.33}$$

In the same way we obtain

$$\begin{aligned} \lambda^5 = \lambda \cdot \lambda^4 &= \lambda[(e_1^3 + 2e_1e_2)\lambda + (e_1^2e_2 + e_2^2)] \\ &= (e_1^4 + 3e_1^2e_2 + e_2^2)\lambda + (e_1^3e_2 + 2e_1e_2^2) \ . \end{aligned} \tag{6.34}$$

Finally,

$$C(\lambda) = C_0\lambda^2 + C_1\lambda + C_2 = (C_0e_1 + C_1)\lambda + (C_0e_2 + C_2) \ . \tag{6.35}$$

By inserting these expressions into equation (6.31), we see that the resulting equation splits into two parts, according to the "surviving" powers of λ:

$$\begin{aligned} \lambda: &\quad (f_1^2e_1 + 2f_1f_2) - (e_1^4 + 3e_1^2e_2 + e_2^2) = 2(C_0e_1 + C_1) \\ 1: &\quad (e_1^2 + e_2 + e_2^2) - (e_1^3e_2 + 2e_1e_2^2) = 2(C_0e_2 + C_2) \ . \end{aligned} \tag{6.36}$$

This method allows to reduce the proof of the separability of the Hamilton–Jacobi equation(s) to the procedure of checking that explicitly known functions identically coincide on the manifold.

We end our discussion of the separability at this point. Our aim was simply to introduce the method of Poisson pairs, and to show by means of concrete examples how it can be profitably used to define and solve special classes of integrable Hamiltonian equations. We hope that the examples discussed in these lectures might be successful in giving at least a feeling of the nature and the potentialities of this method.

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